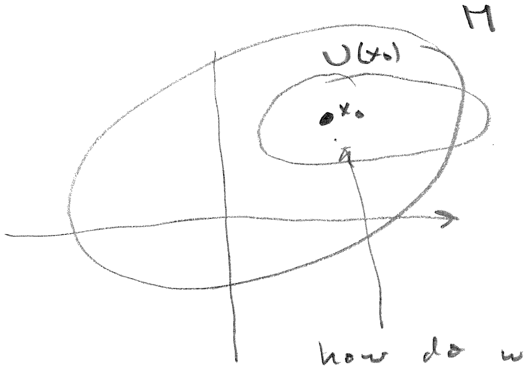
Stability via Liapunov functions



how do we proove stability (if it is true)?

Exercise: 6.15:

$$\dot{x} = x - y - x(x^2+y^2) + \frac{xy}{\sqrt{x^2+y^2}}$$

$$\dot{y} = x + y - y(x^2+y^2) - \frac{x^2}{\sqrt{x^2+y^2}}$$

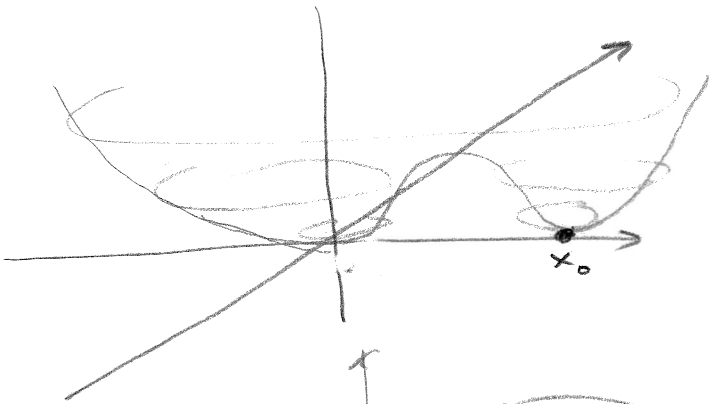
Def Let $L : U(x_0) \to \mathbb{R}$ satisfy

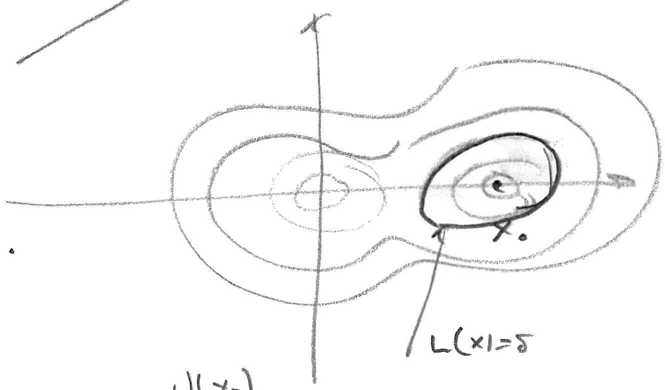
1) $L(x_0) = 0$
2) $L(x) \geq 0$, > 0 if $x \neq x_0$
3) If $t_1 > t_0$, $\phi(t_1) \in U(x_0)$, $\phi(t_0) \in U(x_0)$ $(\notin \{x_0\})$

$\Rightarrow L(\phi(t_1)) < L(\phi(t_0))$

Def

S_δ is the connected component of $\{x \in U(x_0) \mid L(x) \leq \delta\}$ such that $x_0 \in S_\delta$

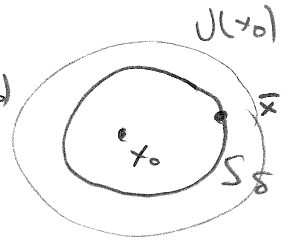




Lemma If S_δ is closed, it is positively invariant.

Proof If $\phi(t)$ leaves S_δ at $\bar{x}$ there is a ball $B_r(\bar{x}) = \{x : |x - \bar{x}| < r\} \subset U(x_0)$ such that $\phi(t_0) = \bar{x}$,

$\phi(t_0+\varepsilon) \in U(x_0) \setminus S_\delta$



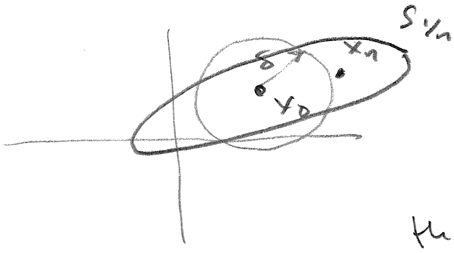
$\Rightarrow L(\phi(t_0+\varepsilon)) < L(\phi(t_0)) \Rightarrow L$ not Liapunov

Lemma For all $\delta > 0$ there is $\varepsilon > 0$ such that

1) $S_\varepsilon \subseteq B_\delta(x_0)$ and

2) $B_\varepsilon(x_0) \subseteq S_\delta$

Proof If 1) is false, then for all $n \geq 1$, there is $x_n \in S_{1/n}$ but $|x_n - x_0| > \delta$. We may take x_n such that $|x_n - x_0| = \delta$, and the sphere $\{x : |x - x_0| = \delta\}$ is compact $\Rightarrow$



there is $\{x_{n_j}\}$ so that $x_{n_j} \to y$ with $|y - x_0| = \delta$. But $L(x_n) \leq \frac{1}{n}$ and L is cont $\Rightarrow$ $L(y) = 0$, contradicting the fact that $L(x) = 0$ only at $x = x_0$.

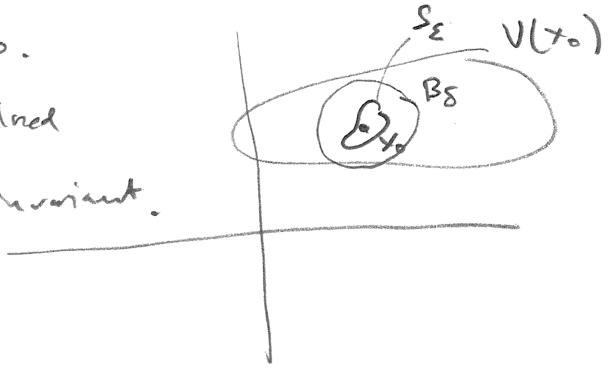
If 2) is false there is $\{x_n\}$ such that $|x_n - x_0| \leq \frac{1}{n}$ and $L(x_n) \geq \delta$.

Then $\delta \leq L(x_n) \to 0$ because $x_n \to x_0$ and $L(x_0) = 0$.

This implies:

Theorem: If x_0 is a fixed point of f and there is a Lyapunov function, then x_0 is stable.

Proof Take a (small) neighborhood $V(x_0)$ of x_0.



S_ε is closed and hence positively invariant.

Example of Lyapunov

$$\begin{cases} \dfrac{dx}{dt} = -3x^3 - y \\ \dfrac{dy}{dt} = x^5 - 2y^3 \end{cases}$$

Show that $(0,0)$ is an asymptotically stable fixed point.

Common idea: try a Liapunov function of the form

$$V(x,y) = a x^{2m} + b y^{2n} \qquad (a, b > 0)$$

$$\frac{d}{dt} V(x(t), y(t)) = 2ma\, x^{2m-1} \dot{x} + 2nb\, y^{2n-1} \dot{y}$$

$$= 2ma\, x^{2m-1} \left(-3x^3 - y \right) + 2nb\, y^{2n-1} \left(x^5 - 2y^3 \right)$$

$$= -6ma\, x^{2m+2} - 2ma\, x^{2m-1} y + 2nb\, y^{2n-1} x^5$$

$$- 4nb\, y^{2n+2}.$$

Let $2m-1 = 5$ $\quad m = 3$

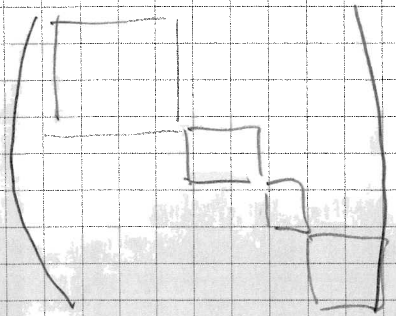
$2n-1 = 1$ $\quad n = 1$

$$\Rightarrow \frac{dV}{dt} = -18a\, x^8 \underbrace{- 6a x^5 y + 2b\, y x^5}_{=0 \text{ if (e.g.) } a=1,\ b=3} - 4nb y^4$$

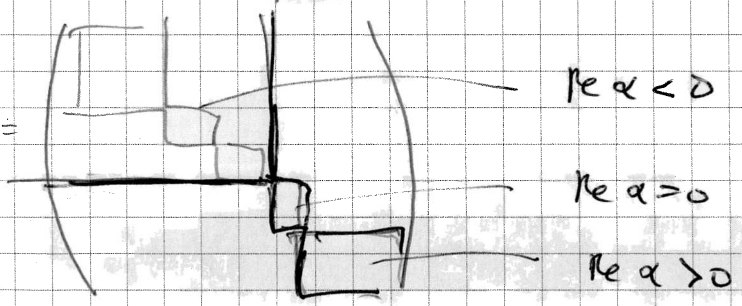
So $V(x,y) = x^6 + 3y^2$ is a strict Liapunov function $\rightarrow$ $(0,0)$ is a stable fixed point.

Back to linear equations (temporarily)

$\dot{x} = Ax$ and in Jordan form

$A =$  where $\square = \begin{pmatrix}\alpha_j & 1 & & 0\\ & \ddots & \ddots & \\ & & \ddots & 1\\ 0 & & & \alpha_j\end{pmatrix}$.

Def $E^+(e^A) = \bigoplus_{\mathrm{Re}\,\alpha<0} N\left((A-\alpha_j)^{q_j}\right)$.

If $A =$ 

when the blocks are ordered so that Re $\alpha<0$ are as in the figure then

$$E^+(e^A) = \mathrm{span}\left(\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix}, \dots, \begin{pmatrix}0\\0\\1\\0\\\vdots\end{pmatrix}\right)$$

We define $E^-(e^A)$ similarly.

and $E^0(e^A)$ too.

So $\quad x \in E^+ \iff e^{tA}x \to 0$ when $t\to\infty$

$\quad x \in E^- \iff e^{-tA}x \to 0$ when $t \to \infty$

Def A system is <u>hyperbolic</u> if all eigenvalues to A satisfy $\operatorname{Re}\alpha \neq 0$.

Theorem Consider $\dot{x} = Ax$ where the eigenvalues are α_j with algebraic and geometric multiplicities a_j, g_j.

- Then $\dot{x} = Ax$ is stable (globally)

  $\Leftrightarrow$ $\operatorname{Re}\alpha_j \leq 0$ and $a_j = g_j$ when $\operatorname{Re}\alpha_j = 0$

- $\dot{x} = Ax$ is asymptotically stable if $\operatorname{Re}\alpha_j < 0$

Then there is a constant α, $\alpha < \min\{-\operatorname{Re}\alpha_j\}$ such that

$$\| e^{tA} \| \leq C e^{-t\alpha}$$

Theorem The stable and unstable subspaces E^+, E^- are invariant.

$$X_0 \in E^+ \Rightarrow |e^{tA} x_0| \leq C e^{-t\alpha} |x|$$

$$X_0 \in E^- \Rightarrow |e^{tA} x_0| \leq C e^{-t\alpha} |x|$$