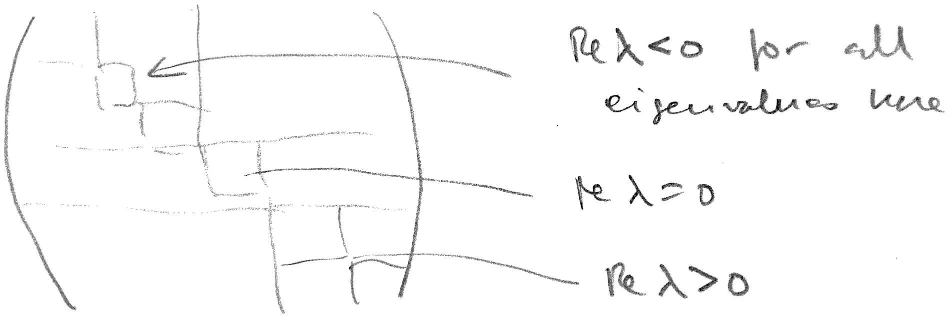
Back to linear systems

$\dot{x} = Ax$ Fixed point: $x = 0$.

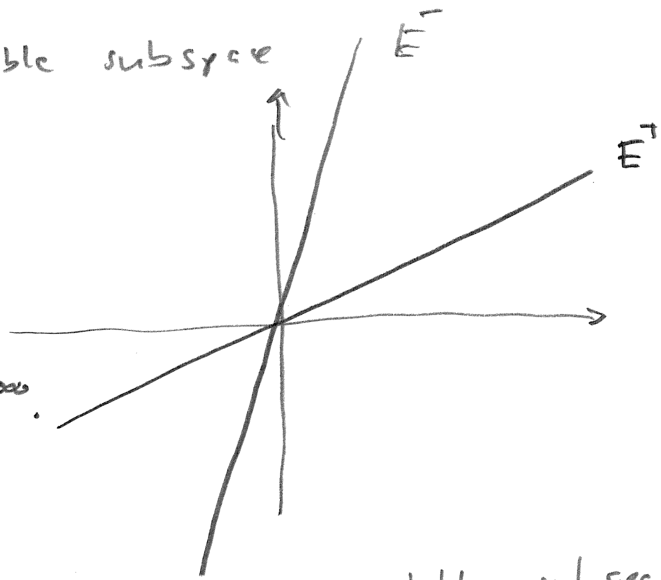
Assume A in Jordan form:

$A =$ 

Def $E^+(e^A) = \bigoplus_{\mathrm{Re}\,\alpha_j < 0} \mathcal{N}\left((A-\alpha_j)^{a_j}\right)$ stable subspace

(α_j eigenvalue to a Jordan block)

$E^-(e^A)$ unstable subspace



$x \in E^+ \Rightarrow e^{tA}x \to 0$

and there is $\alpha > 0$ such that

$e^{\alpha t} |e^{tA}x| \to 0$ when $t \to \infty$.

$x \in E^- \Rightarrow e^{-tA}x \to 0$

and there is $\alpha > 0$ s.t.

$e^{\alpha t} |e^{-tA}x| \to 0$ when $t \to \infty$

unstable subspace

$E^0(e^A) = \bigoplus_{\mathrm{Re}\,\alpha_j = 0} \mathcal{N}\left((A-\alpha_j)^{a_j}\right)$ center subspace.

Def A system is <u>hyperbolic</u> if all eigenvalues satisfy $\operatorname{Re}\alpha \neq 0$.

[why is this the typical case?]

Theorem Consider $\dot{x} = Ax$ where the eigenvalues α_j have algebraic and geometric multiplicity a_j, g_j.

Then $x=0$ is stable $\Leftrightarrow$ $\operatorname{Re}\alpha_j \leq 0$ and $a_j = g_j$ if $\operatorname{Re}\alpha_j = 0$

asymptotically stable $\Longleftrightarrow$ $\operatorname{Re}\alpha_j < 0$.

There is a constant α such that

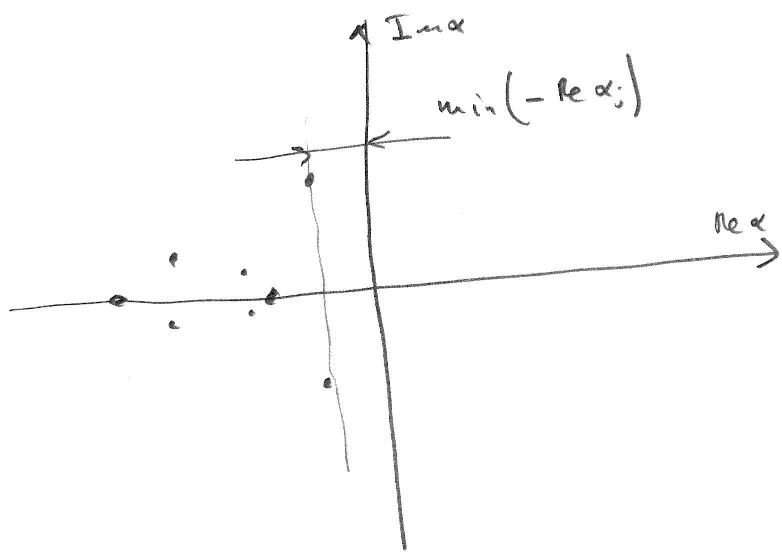
$0 < \alpha < \min\{-\operatorname{Re}\alpha_j\}$

and $\|e^{tA}\| \leq C e^{-t\alpha}$ ($\to 0$ when $t \to \infty$).

Theorem The linear stable and unstable manifolds E^+, E^- are invariant.

and $x_0 \in E^+ \Rightarrow |e^{tA}x_0| < Ce^{-t\alpha}|x_0|$

$x_0 \in E^- \Rightarrow |e^{-tA}x_0| < Ce^{-t\alpha}|x_0|$

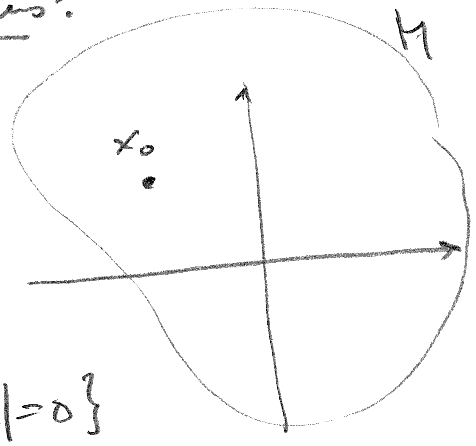


Def $$E^{+,\alpha} = \bigoplus_{-\mathrm{Re}(\alpha_j) > \alpha} \mathcal{N}\left((A-\alpha_j)^{a_j} \right) = E^+\left(e^{A+\alpha}\right)$$

$$E^{-,\alpha} = E^-\left(e^{A-\alpha}\right)$$

Equiv $$E^{+,\alpha}(e^A) = \{ x : \lim_{t\to\infty} e^{\alpha t} \| e^{tA} \| = 0 \}$$

What to do about nonlinear systems?



$\dot{x} = f(x)$ with $f(x_0) = 0$.

Def The stable set is

$$W^+ = \{ x \in M : \lim_{t\to\infty} |\Phi_t(x) - x_0| = 0 \}$$

The unstable set is

$$W^- = \{ x \in M : \lim_{t\to -\infty} |\Phi_t(x) - x_0| = 0 \}$$

Def Let $f(x) = A(x-x_0) + g(x-x_0)$

where $g(z) = o(|z|)$.

x_0 is said to be hyperbolic if A is.

Question: What do W^+ and W^- look like?

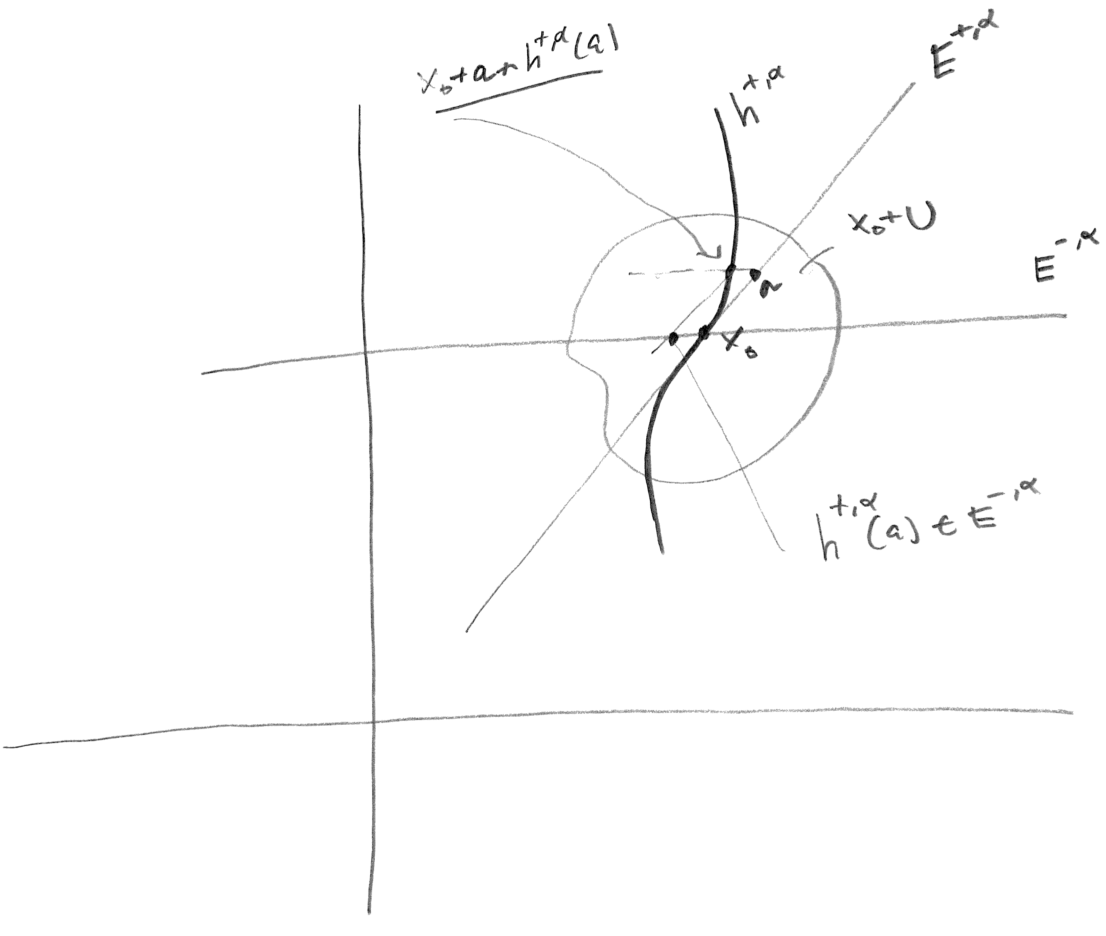
Theorem Assume that $f \in C^k(\mathcal{U})$ and that $f(x_0)=0$, $f(x) = A(x-x_0) + g(x-x_0)$ where $g(z) = o(|z|)$. Take $\alpha > 0$ and assume $A+\alpha I$ is hyperbolic. Then there is $U(x_0) = x_0 + U$ and $h^{+,\alpha} \in C^k(E^{+,\alpha} \cap U, E^{-,\alpha})$ such that

$$M^{+,\alpha}(x_0) \cap U(x_0) = \{x_0 + a + h^{+,\alpha}(a),\ a \in E^{+,\alpha} \cap U\}$$

$M^{+,\alpha}$ is tangent to $E^{-,\alpha}$ at x_0.

Proof: A contraction mapping argument, not so easy.

What does the theorem mean?



Diffeomorphisms

Discrete dynamical systems:

$$x_{n+1} = f(x_n) \qquad f: \mathbb{R}^n \to \mathbb{R}^n \qquad (f: M \to M)$$

Def f is a C^r-map if it r times cont differentiable

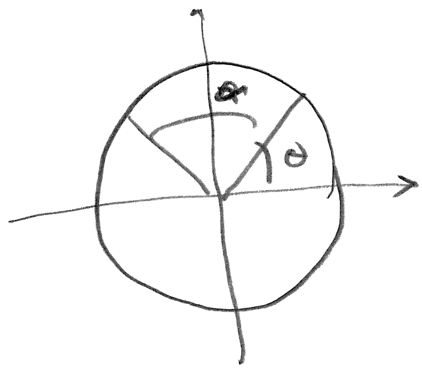
f is a C^k-diffeomorphism if it is a bijection and f and f^{-1} are C^k.

f is a homeomorphism if it is a bijection and f and f^{-1} are C^0.

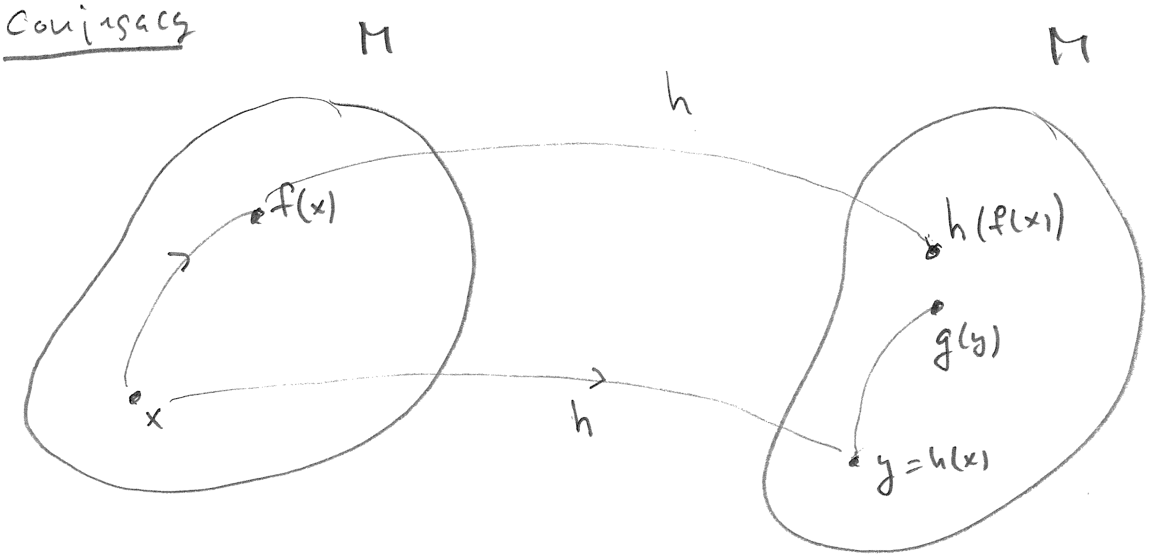
Ex Diffeomorphisms on the circle:

$R_\alpha: \theta \mapsto \theta + \alpha \pmod 1$

rotation with angle α



Conjugacy



Can it happen that $g(h(x)) = h(f(x))$?

Def Two diffeomorphisms $f, g : M \to M$ are <u>topologically conjugate</u> if there is a homeomorphism $h : M \to M$ such that

$$h \circ f = g \circ h \quad \text{for all } x \in M$$

For flows $\Phi(t,x)$:

Def Φ_t and Ψ_t are topologically conjugate if there is $h : M \to M$ such that

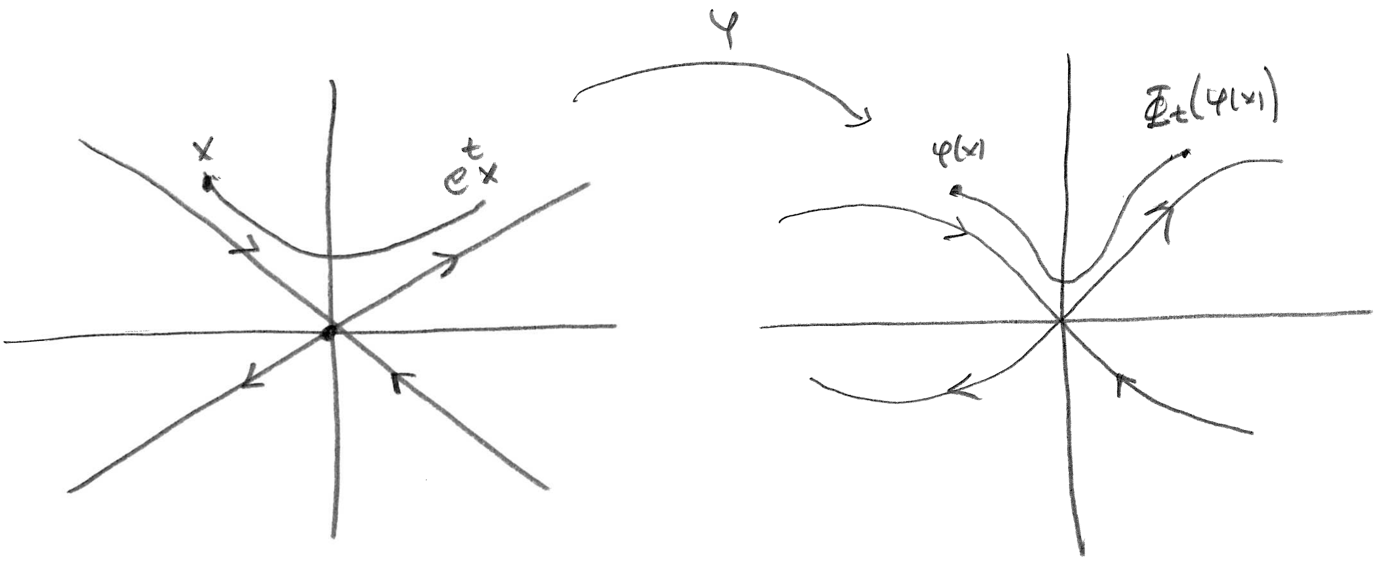
$$h(\Phi_t(x)) = \Psi_t(h(x)) \quad \text{for all } t.$$

Def The flows Φ_t and Ψ_t are topologically equivalent if there is a homeomorphism $h : M \to M$ such that h maps orbits of Φ_t onto orbits of Ψ_t.

| Which of these is a stronger statement? |

<u>Theorem</u> Assume f is C^1, and that $f(0)=0$ is a hyperbolic fixed point. Let $\Phi(t,x)$ be the flow of $\dot{x} = f(x)$ and let $A = df(0)$ $(= f'(0))$ (so that $f(x) = Ax + g(x)$ with $g(x) = o(|x|)$).
There is a homeomorphism $\varphi(x) = x + h(x)$ such that (locally)

$$\varphi \circ e^{tA} = \Phi_t \circ \varphi$$



<u>Question</u> Have we seen other examples of topologically conjugate flows?

The straightening of of vector field theorem.