Inhomogeneous equation

$$\begin{cases} \dot{x} = A(t)x + g(t) \\ x(t_0) = x_0 \end{cases} \qquad \begin{array}{l} A \in C(I, \mathbb{R}^{n\times n}) \\ g \in C(I, \mathbb{R}^n) \end{array}$$

Ansatz Look for a solution of the form

$$x(t) = \Pi(t,t_0)\, c(t) \qquad \text{where } c(t_0) = x_0$$

$$\dot{x}(t) = A(t)\Pi(t,t_0)c(t) + \Pi(t,t_0)\dot{c}(t)$$

$$Ax(t) = A(t)\Pi(t,t_0)c(t)$$

$$\Rightarrow \quad A(t)\Pi(t,t_0)c(t) + \Pi(t,t_0)\dot{c}(t) =$$

$$= A(t)\Pi(t,t_0)c(t) + g(t)$$

$$\Rightarrow \quad \Pi(t,t_0)\dot{c}(t) = g(t)$$

$$\Rightarrow \quad \dot{c}(t) = \Pi(t,t_0)^{-1} g(t)$$

$$c(t) = x_0 + \int_{t_0}^{t} \Pi(t_0,s)\, g(s)\, ds$$

$$\Rightarrow \quad x(t) = \Pi(t,t_0)x_0 + \int_{t_0}^{t} \underbrace{\Pi(t,t_0)\Pi(t_0,s)}_{\Pi(t,s)}\, g(s)\, ds$$

Dynamical systems

Recall: a semigroup G acting on a space M, i.e.

a map $T: G\times M \to M$

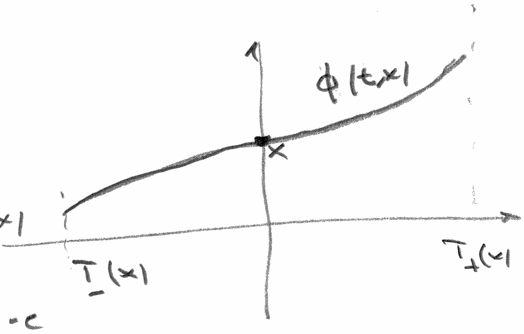
$$(g,x) \mapsto T_g x \quad , \quad T_g \circ T_h = T_{g\circ h}$$

Ex $\quad G = \mathbb{N}_0, \quad T_1: x \to f(x)$, a map.

$$T_k x = \underbrace{f(f(f\ldots f}_{k \text{ times}}(x)\ldots)$$

Ex $\quad G = \mathbb{R} \quad \begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$ with unique solution $\phi_t(x_0)$

$$T_t: x_0 \longmapsto \phi_t(x_0)$$



Def The flow of an ode:

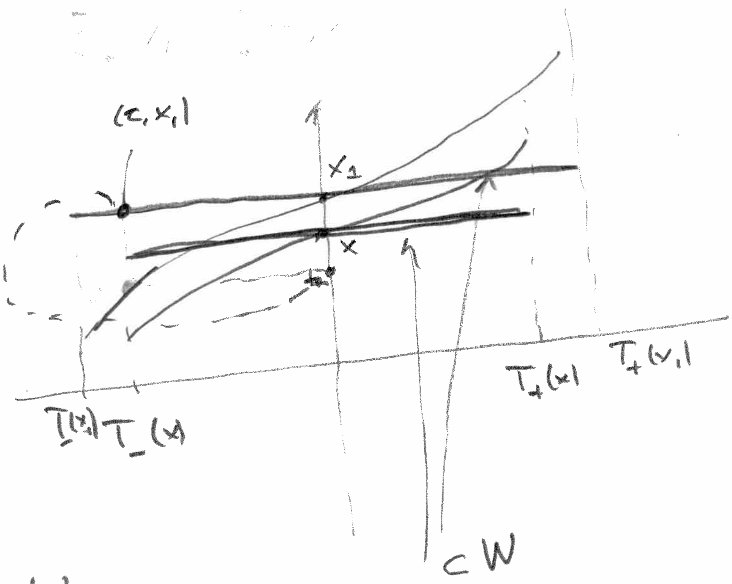
Consider $\begin{cases} \dot{y} = f(y) \\ y(0) = x \end{cases}$ unique solution $\phi(t,x)$

There is a maximal integral curve

defined on a maximal interval $]T_-(x), T_+(x)[$

Let $W = \bigcup_{x\in M} I_x \times \{x\}$

Φ



Let $\Phi: W \to M$

$$(t,x) \to \phi(t,x)$$

Notation: $\Phi_t(x) = \Phi(t,x) = \Phi_x(t)$

[why such notation?]

Theorem Let $f \in C^k(M, \mathbb{R}^n)$ $\quad M \subset \mathbb{R}^n$

For all $x \in M$, there is $I_x \subset \mathbb{R}$, $0 \in I_x$

such that there is a unique maximal integral curve $\Phi(\cdot, x) \in C^k(I_x, M)$.

$W = \bigcup_{x \in M} I_x \times \{x\}$ is open and $\Phi \in C^k(W, M)$

is a local flow on M:

$$\Phi(0,x) = x$$
$$\Phi(s+t, x) = \Phi(s, \Phi(t,x))$$

Proof [what do we need to prove?]

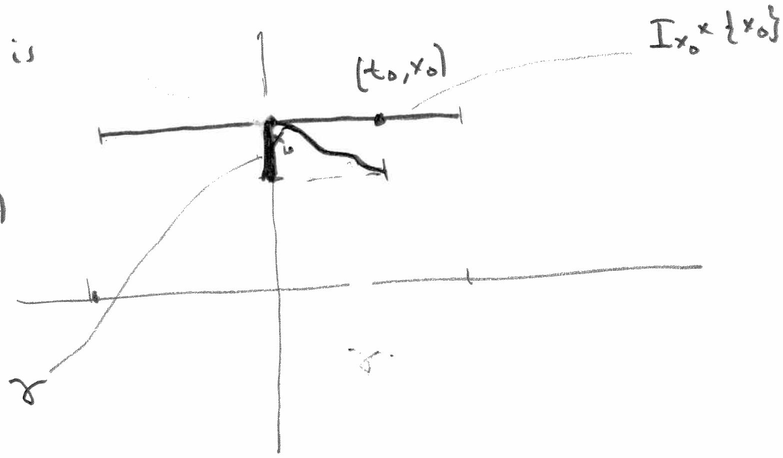
That Φ is open and $\Phi \in C^k(W,M)$.

Fix $(t_0, x_0) \in W$, and let $\gamma = \Phi_{x_0}([0,t_0])$

For any $x \in \gamma$, there is

$(-\varepsilon(x), \varepsilon(x)) \times U(x)$, an open neighborhood of $(0,x)$ such that

$\Phi(t,x)$ is defined in that neighborhood,



γ is compact (why?)

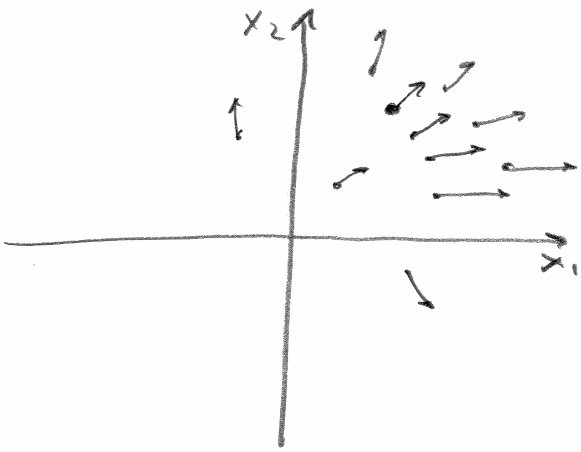
$\Rightarrow$ there is a finite number of $x_1, \ldots, x_m \in \gamma$ such that $\{0\} \times \gamma \subset \bigcup_{i=1}^{m} (-\varepsilon(x_i), \varepsilon(x_i)) \times U(x_i)$

Let $\varepsilon = \min \varepsilon(x_i)$ and $U_0 = \bigcap U(x_i)$ $\Big\} \Rightarrow$ $\Phi(t,x)$ is defined on $(-\varepsilon, \varepsilon) \times U_0$

Phase portraits, vector fields

In $\mathbb{R}^2$: A vector field is a map $\mathbb{R}^2 \to \mathbb{R}^2$:

$f: x \mapsto f(x) \in \mathbb{R}^2$



An integral curve:

A curve

$\Phi(s) \in \mathbb{R}^2, s \in I$

such that this curve is parallel to the vector field:

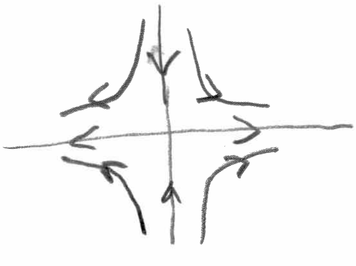
$$\frac{d\Phi}{ds}(s) \parallel f(\Phi(s))$$

A phase portrait: a picture showing

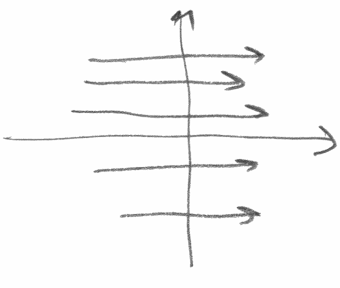
the characteristic qualities of a vector field: fixed points, closed orbits...

Very different if we look at the field locally (close to a point) or globally:

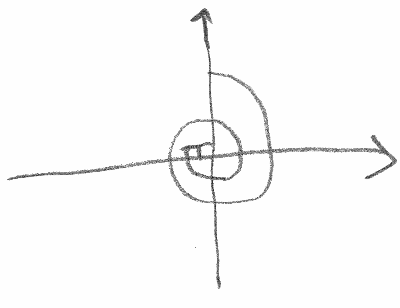
Near A



B



C



<u>Next</u> take $m \in \mathbb{N}$ so large that $\frac{t_0}{m} < \varepsilon$

and let $K : U_0 \to M$
$x \mapsto \Phi_{t_0/m}$. Then $K \in C^k(U_0)$

And $K^{j+1} \in C^k(U_j)$ where $U_j = K^{-j}(U_0) \subset U_0$ is open
[why??].

By definition, $x_0 = K^{-j}\left(\Phi\left(\frac{j}{m} t_0, x_0\right)\right)$

Therefore $x_0 \in U_j \Rightarrow U_j \neq \phi$.

And $\Phi(t,x) = \Phi(t-t_0, \Phi(t_0,x)) =$

$= \Phi(t-t_0, K^m(x))$

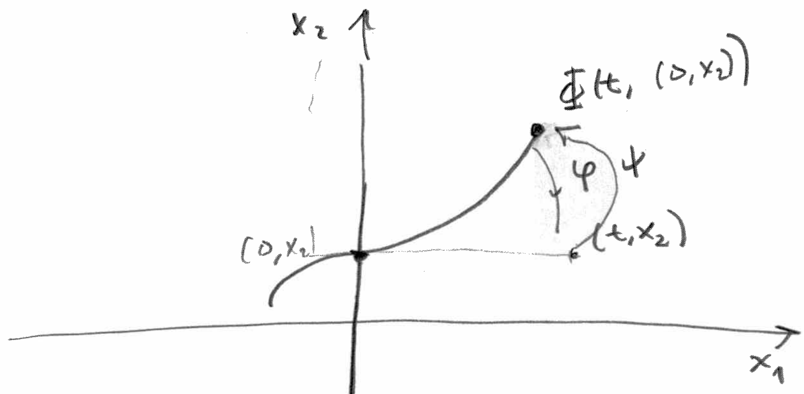
is well defined for all $(t,x) \in (t_0-\varepsilon, t_0+\varepsilon) \times U_m$.

<u>Lemma</u> "straightening out of vector fields"
Assume $f(x_0) \neq 0$. Then there is a local coordinate transfor $y = \varphi(x)$: $\dot{x} = f(x)$
is transformed to

$$\dot{y} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

(so a trivial equation).

<u>Proof</u> Assume $x_0 = 0$, and that $f(0) = (1, 0, \ldots 0)^t$
[why is this always possible?]



Consider a transformation: $\varphi : \Phi(t, (0, x_2 \ldots x_n)) \longmapsto (t, x_2, \ldots x_n)$

so $\varphi = \psi^{-1}$ where

$$\psi(x_1, \ldots, x_n) = \Phi(x_1, (0, x_2, \ldots, x_n))$$

This is well defined near 0.

$$\det \frac{\partial \psi_i}{\partial x_j}\Big|_{x=0} = \det\left(\frac{\partial \Phi}{\partial t}, \frac{\partial \Phi}{\partial x_2} \cdots \frac{\partial \Phi}{\partial x_n}\right)\Big|_{\substack{t=0 \\ x_2=0 \ldots}} =$$

$$= \det\begin{pmatrix} f, & \begin{matrix}0\\1\\ \vdots\\0\end{matrix}, & \begin{matrix}0\\0\\1\\ \vdots\end{matrix}, & \ldots, & \begin{matrix}0\\ \vdots\\ \vdots\\1\end{matrix}\end{pmatrix} = 1$$

$\boxed{\text{why?}}$ $\Rightarrow$ Therefore ψ is a local diffeomorphism.

Let $\quad \psi(y) = x$

$y = \psi^{-1}(x) \qquad \dfrac{\partial \psi_i}{\partial y_j}\dot{y}_j = f_i$

$$\begin{pmatrix} f(y) & ? \end{pmatrix}\dot{y} = f$$

$$\Rightarrow \dot{y} = \begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix} \qquad [\text{why?}]$$

There is a note about the implicit function theorem