A little topology

This is a collection of definitions and results from topology, which are needed for the ode course. The notes are certainly not intended to be part of a course in topology, there are many good books to study then – Simmons: "Introduction to topology and modern analysis" is one example.

Let M be a set of points (in most cases M will be $\mathbb{R}^n$ or a subset of $\mathbb{R}^n$).

A *metric* on M is a function $\rho : M \times M \to \mathbb{R}$ such that for all $x, y, z \in M$

$$\begin{aligned}
&\rho(x,y) = \rho(y,x) \\
&\rho(x,y) \geq 0 \\
&\rho(x,y) = 0 \Leftrightarrow \ x = y \\
&\rho(x,y) \leq \rho(x,z) + \rho(z,y)
\end{aligned}$$

An (open) ball of diameter r around $x \in M$ is the set

$$B_r(x) = \{ \ y \in M \ : \ \rho(y,x) < r \ \}$$

A subset of M, $U \subset M$ is *open* if for each $x \in U$, there is a ball $B_r(x)$ such that

$$B_r(x) \subset U$$

A subset of M, $K \subset M$ is *closed* if it contains all limit points, *i.e.* if $x_1, x_2, ... \in K$ and $x_k \to x$ when $k \to \infty$, then $x \in K$. The complement ($K^c = M \setminus K = \{x \in M \ : \ x \notin K\}$) of an open set is closed.

A *neighbourhood* of x is a set U that contains x and such that x is an interior point of U. Another way to say this is that there is an open set $V \subset U$ such that $x \in V$. Note that it is not uncommon to use the word neighbourhood for *open neigbourhood*, i.e. an open set that contains x.

Let A be a subset of M. The *limit set* of A is the set of

$$\{x \in M \ : \ \text{there is a sequence } \ x_k \in A, k = 1, 2, 3..., \lim_{k\to\infty} x_k = x\}$$

This is a closed set, and it is the *closure* of A. It is denoted $\bar{A}$.

A subset $C \subset \mathbb{R}^n$ is *compact* if it is closed and bounded. For us the most important property of compact sets is the following: Let $x_k \in C, k = 1, 2, 3...$ be a sequence of points in C. Then there is a subsequence $x_{k_j}, j = 1...\infty$ that is convergent,

$$\lim_{j\to\infty} x_{k_j} = x \in C \, .$$

Note that a sequence x_k may contain many convergent subsequences converging to different points in C, but at least one.
The meaning of compact is in fact this: if a B is a subset of C that contains infinitely many points, then this infinite set of points must concentrate at least at one point in C, there is not enough space to keep all these points spread out.
If M is not $\mathbb{R}^n$ it there may be closed and bounded sets that are not compact, and then another definition is needed. In fact, if M is metric, *i.e.* there is a metric $\rho(x, y)$,

then a set $C \subset M$ is said to be compact if every infinite sequence of points contains a convergent subsequence. So the important result about closed and bounded sets in $\mathbb{R}^n$ is taken to be the definition of compactness.
There are other definitions that can be used for topological spaces (this term needs a definition) M that are not metric.

Continuous functions

Let M and N be metric spaces, and let $f : M \to N$ be a function from M to N. The function f is said to be *continuous* if when $x, x_1, x_2, x_3, ... \in M$ and $\lim_{k\to\infty} x_k = x$, then $\lim_{k\to\infty} f(x_k) = f(x)$. A different, equivalent definition is the following: Let $U \subset N$ be any open set and define $f^{-1}(U)$ to be the set

$$f^{-1}(U) = \{x \in M \;:\; f(x) \in U\}$$

The function f is continuous if $f^{-1}(U)$ is open. We prove that this second definition implies the first: take $x \in M$ and let $y = f(x) \in N$. Let $x_k, k = 1, 2, 3...$ be a sequence in M with $\lim_{k\to\infty} x_k = x$. We must prove that $\lim_{k\to\infty} f(x_k) = y$. To this end, take a ball $B_r(y) \subset N$ with center at y, and consider the shrinking sequence of balls $B_{r/n}(y)$. These are open sets, each of them containing y and according to the second definition of continuous, all the sets $U_n = f^{-1}(B_{r/n})$ are open, and $x \in U_n$ for all n. And because the sequence x_k converges to x we must have $x_k \in U_n$ for all sufficiently large k. We have $U_m \subset U_n$ if $m > n$, and therefore ”sufficiently large” increases with n. Now chose $\varepsilon > 0$ arbitrarily small, and take n so large that $r/n \le \varepsilon$. Next chose K_n so large that $x_k \in U_n$ for all $k \ge K_n$. But $U_n = f^{-1}(B_{r/n}(y))$ so therefore $f(x_k) \in B_{r/n}(y)$ for $k \ge K_n$, and this means that $\rho_N(f(x_k), y) < \varepsilon$ for $k > K_n$. (I have written ρ_N for the metric in N). Hence the sequence $f(x_k)$ converges to y when $k \to \infty$ and the proof is ready.
Next, suppose that $x_k \to x$ implies $f(x_k) \to f(x)$. We want to prove then that for any open $U \subset N$, $f^{-1}(U)$ is open. If $f^{-1}(U)$ is not open, there is $x \in f^{-1}(U)$ and a sequence of points $x_k \notin f^{-1}(U)$ such that $\lim_{k\to\infty} x_k = x$. But then $\lim_{k\to\infty} f(x_k) = f(x) \in U$, and because U is open there is a ball $B_r(f(x)) \subset U$ and then $f(x_k) \in U$ for all sufficiently large k. But then, for k sufficiently large, $x_k \in f^{-1}(U)$, which is a contradiction. Hence we conclude that $f^{-1}(U)$ must be open.
The second characterization of continuous functions can be used also in topological spaces that are not metric.

Differentiable functions $M \to M$ and the implicit function theorem

If $f \in C^1(\mathbb{R})$, and $f'(x_0) \neq 0$, there is an interval $]x_0 - \varepsilon, x_0 + \varepsilon[$ such that $f'(x) \neq 0$ for all x in this interval. We may assume that $f'(x) > 0$ in the interval, becuase the opposite case works in the same way. Then $f(x)$ is strictly monotonously increasing in the interval, and we know that there is an inverse function:

$$y = f(x) \quad \Leftrightarrow \quad x = f^{-1}(y)$$

for all $x \in]x_0 - \varepsilon, x_0 + \varepsilon[$. The inverse function is also differentiable, and

$$\frac{d}{dy} f^{-1}(y) \quad = \quad \frac{1}{f'(x)} \quad (y = f(x))$$

The relateion between the derivatives of f and f^{-1} follows by the chain rule. Set $g(y) = f^{-1}(y)$, to avoid cumbersome notation. Then

$$\begin{aligned} x = g(f(x)) \quad &\Rightarrow \quad 1 = \frac{d}{dx} g(f(x)) = g'(f(x))f'(x) \\ &\Rightarrow \quad g'(f(x)) = \frac{1}{f'(x)} \end{aligned}$$

The same argument can be carried out for functions $f \in C'(M, M)$, where $M \subset \mathbb{R}^n$.. Suppose first that the function $f : M \to M$ is differentiable and that there is an inverse function g such that for all $x \in M \subset \mathbb{R}^n$,

$$x \quad = \quad g(f(x))$$

Then, differentiating with respect to x we get

$$I \quad = \quad g'(f(x))f'(x)$$

i.e. exactly the same expression as in the one dimensional case, except that in the left hand side, we have the $n \times n$ identity matrix, and

$$g'(y) \quad = \quad \begin{pmatrix} \frac{\partial g_1}{\partial y_1} & \frac{\partial g_1}{\partial y_1 2} & \cdots & \frac{\partial g_1}{\partial y_n} \\ \frac{\partial g_2}{\partial y_1} & \frac{\partial g_2}{\partial y_1 2} & \cdots & \frac{\partial g_2}{\partial y_n} \\ & & \cdots & \\ \frac{\partial g_n}{\partial y_1} & \frac{\partial g_n}{\partial y_1 2} & \cdots & \frac{\partial g_n}{\partial y_n} \end{pmatrix}$$

and similary for $f'(x)$. And if the matrix $f'(x)$ is invertible, then

$$g'(y) \quad = \quad f'(x)^{-1}$$

where in the righthand side we mean the matrix invers of $f'(x)$. The *inverse function theorem* states that, just like in the one-dimensional case, if $det(f'(x_0)) \neq 0$ for some $x_0 \in M$, then there is an open set U which contains x_0 such that $det(f'(x)) \neq 0$ for all $x \in U$, and such that the function $f(x)$ has an inverse g that is defined on U, and the stated formula for g' is valid.