9.3 Bifurcations

Cycles and equilibrium points are distinctive features of orbital portraits of planar autonomous systems. If the rate functions of the system are changed slightly, you expect the new portrait to resemble the old: a cycle might contract, an equilibrium point shift, a spiral tighten or loosen, but the dominant features of the portrait would remain. This seems reasonable, but it is not always true. As a parameter changes, an equilibrium point may suddenly appear and spawn a second equilibrium point, or a stable equilibrium point may destabilize and eject an attracting limit cycle. These are *bifurcations*. In the field of ODEs the word "bifurcation" has come to mean any marked change in the structure of the orbits of a system (usually nonlinear) as a parameter passes through a critical value. We describe several kinds of bifurcation that occur in planar autonomous systems.

Saddle-Node Bifurcation

The simplest bifurcations involve the appearance, disappearance, or splitting of an equilibrium point as a parameter changes. One example is given here; other types appear in Problem 1. Some of these bifurcations are direct extensions to planar systems of bifurcations of first-order ODEs described in Section 2.9 and its problem set.

EXAMPLE 9.3.1

An Example of a Saddle-Node Bifurcation

The system

$$x' = c + x^2, \qquad y' = -y \tag{1}$$

has no equilibrium point if the parameter c is positive. See Figure 9.3.1 for orbits if $c = 1$. If $c = 0$, a strange hybrid of saddle point and stable node suddenly appears at the origin, resembling a node on the left, a saddle on the right (Figure 9.3.2). This is a *saddle-node* equilibrium point. It is not an elementary equilibrium point like those in Section 6.5 because $\lambda = 0$ is an eigenvalue of the Jacobian matrix of system (1) at the origin. As c decreases below zero, the origin splits into two equilibrium points $(\pm\sqrt{-c}, 0)$, one a saddle point and the other an asymptotically stable node. These points move in opposite directions away from the origin as c decreases (Figure 9.3.3).

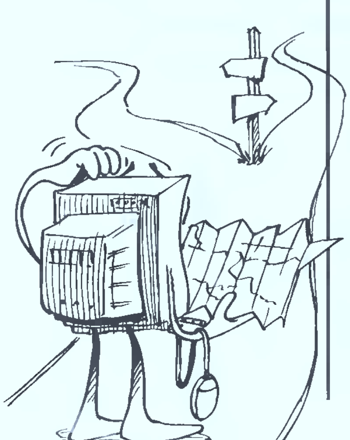


Figure 9.3.4 shows the *bifurcation diagram* for system (1). The diagram plots the x-coordinates of the equilibrium points as functions of c. The solid curve denotes the x-coordinate of the stable equilibrium point and the dashed curve denotes the x-coordinate of the unstable equilibrium point. This *bifurcation diagram* describes the evolution of a system's orbits as a parameter changes. The saddle-node bifurcation is an example of a tangent bifurcation (see Section 2.9).

Now let's take a look at a very different kind of bifurcation.

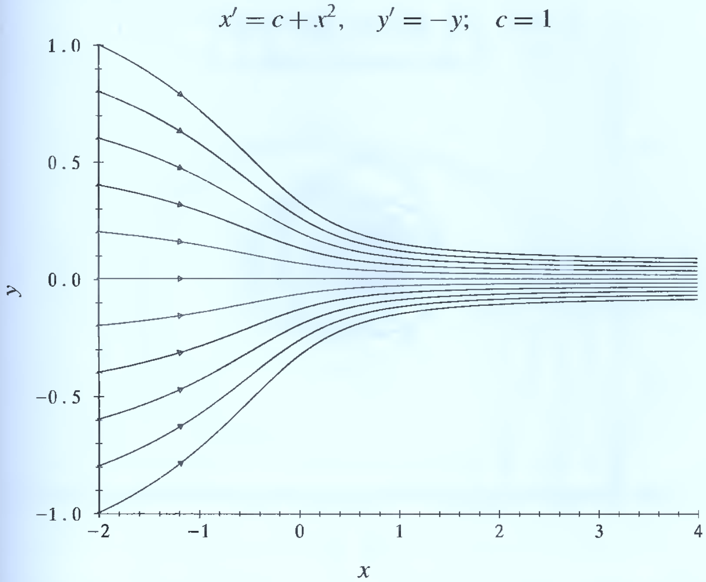


FIGURE 9.3.1 No equilibrium point before the saddle-node bifurcation: $c = 1$ (Example 9.3.1).

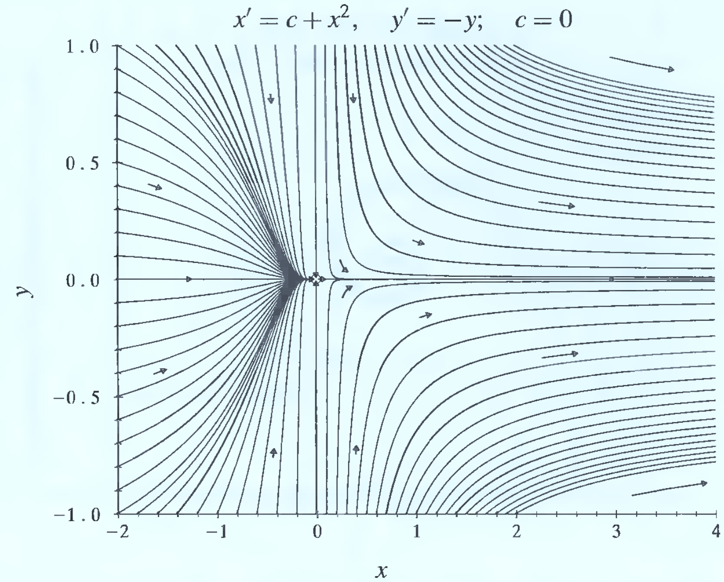


FIGURE 9.3.2 Saddle-node equilibrium point at bifurcation: $c = 0$ (Example 9.3.1).

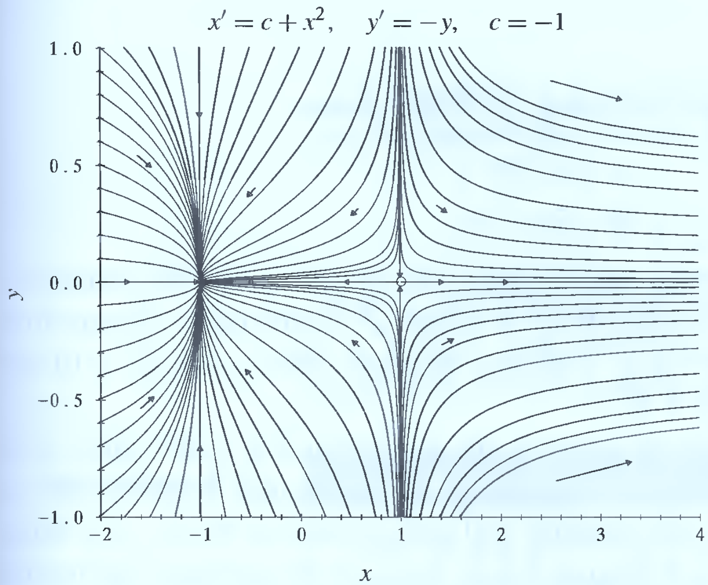


FIGURE 9.3.3 Saddle and node after the saddle-node bifurcation: $c = -1$ (Example 9.3.1).

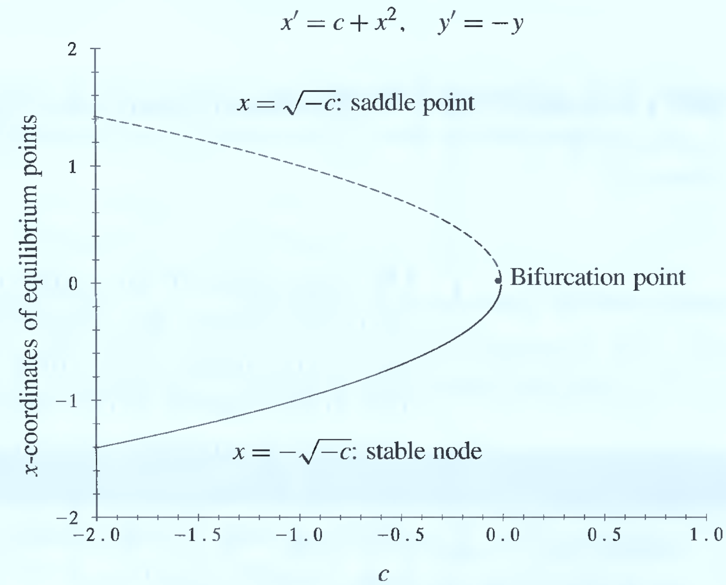


FIGURE 9.3.4 The saddle-node bifurcation diagram (Example 9.3.1).

The Hopf Bifurcation

In the next example an equilibrium point expands into a limit cycle as a parameter changes.

EXAMPLE 9.3.2

Bifurcation to a Limit Cycle

For all values of the parameter c, the origin is an equilibrium point of the system

$$\begin{aligned} x' &= cx + 5y - x(x^2 + y^2) \\ y' &= -5x + cy - y(x^2 + y^2) \end{aligned} \qquad (2)$$

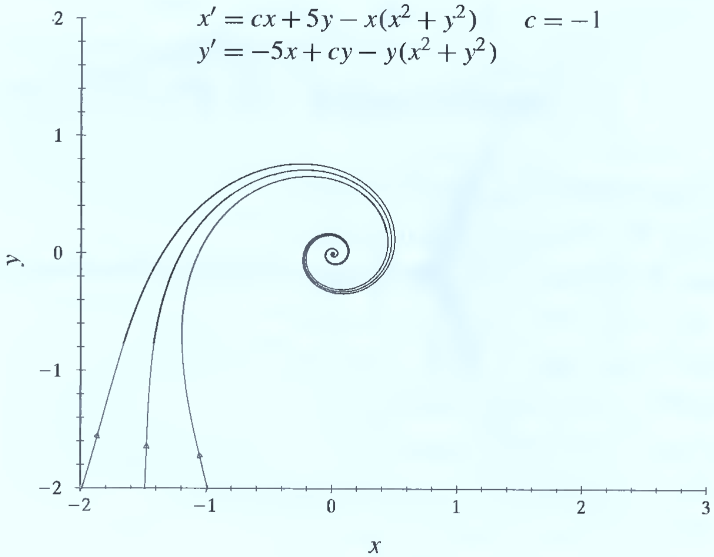


FIGURE 9.3.5 Asymptotic stability before the Hopf bifurcation: $c = -1$ (Example 9.3.2).

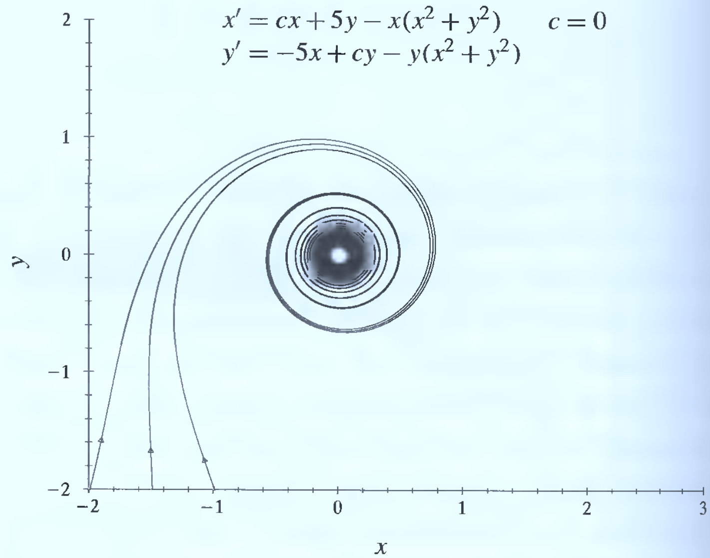


FIGURE 9.3.6 Asymptotic stability at the Hopf bifurcation value: $c = 0$ (Example 9.3.2).

☞ Linearizations are described in Section 8.2.

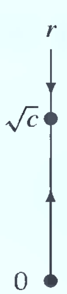
The linearization at the origin for system (2) is the system

$$\begin{aligned} x' &= cx + 5y \\ y' &= -5x + cy \end{aligned} \tag{3}$$

The eigenvalues of the matrix of the linear system (3) are $c \pm 5i$. As the parameter c increases through zero, the equilibrium point of system (3) at the origin changes from an asymptotically stable spiral point ($c < 0$), to a neutrally stable center ($c = 0$), and then to an unstable spiral point ($c > 0$).

For the nonlinear system (2) the origin is an asymptotically stable spiral point if $c < 0$ (Figure 9.3.5), because the eigenvalues of the coefficient matrix of the linearized system (3) are complex with negative real parts (Theorem 8.2.1). The origin is an unstable spiral point if $c > 0$ (Figure 9.3.7), because the complex eigenvalues have positive real parts. Something different happens at $c = 0$: the origin is still an asymptotically stable spiral point of system (2), but orbits spiral inward very slowly (Figure 9.3.6). As the value of c increases through zero, the origin destabilizes and emits an attracting circular limit cycle of radius $\sqrt{c}$ (Figure 9.3.7). This behavior is easier to understand if you write system (2) in polar coordinates:

$$r' = r(c - r^2), \qquad \theta' = -5 \tag{4}$$



From system (4), you see that the circle $r = \sqrt{c}$ is indeed an orbit if $c > 0$ (the state line for r is in the margin). The system traverses the circle clockwise (since θ' is negative) and attracts all other nonconstant orbits since r' is positive if $0 < r < \sqrt{c}$ and negative if $r > \sqrt{c}$.

Figure 9.3.8 shows a bifurcation diagram for system (2) and the equivalent system (4). The horizontal c-axis is solid for $c < 0$, because the origin is an asymptotically stable equilibrium point, and dashed for $c > 0$, because the origin is unstable for positive c. The solid curve $r = \sqrt{c}$ depicts the amplitude of the attracting limit cycle.

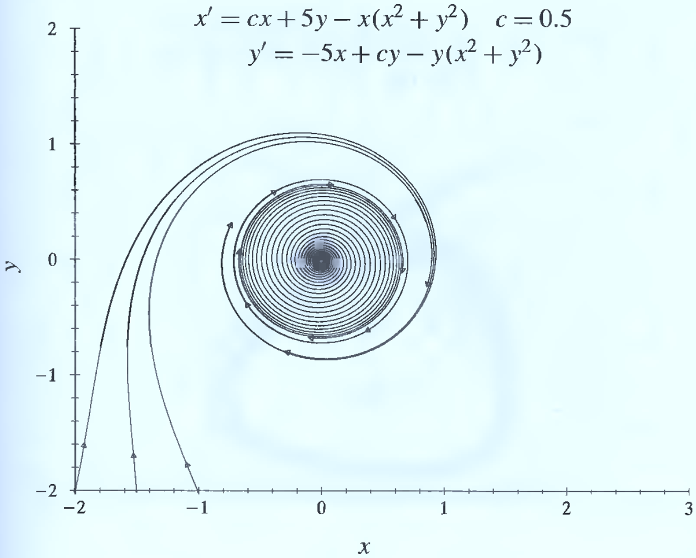


FIGURE 9.3.7 Attracting limit cycle $r = \sqrt{c}$ after the Hopf bifurcation: $c = 0.5$ (Example 9.3.2).

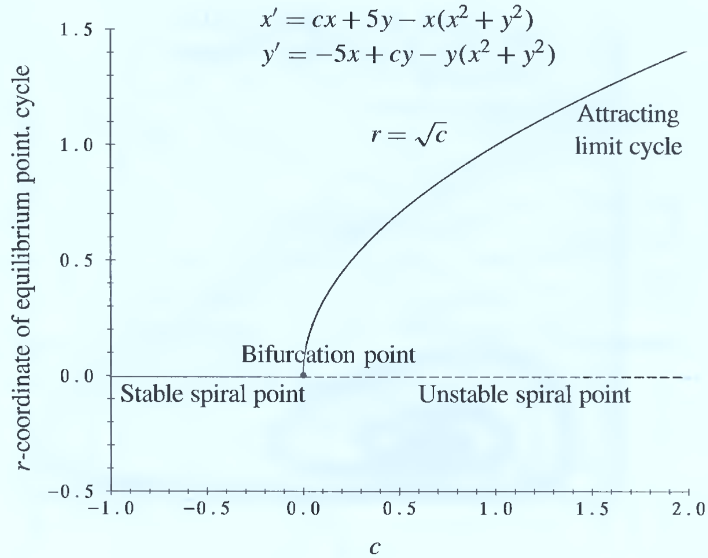


FIGURE 9.3.8 Hopf bifurcation diagram (Example 9.3.2).

Example 9.3.2 shows an example of a general mechanism for creating limit cycles that the Dutch mathematician Eberhard Hopf discovered. Look at the system

$$\begin{aligned} x' &= \alpha(c)x + \beta(c)y + P(x, y, c) \\ y' &= -\beta(c)x + \alpha(c)y + Q(x, y, c) \end{aligned} \tag{5}$$

where P and Q are at least second order in x, y and twice continuously differentiable in x, y, and c, and $\alpha(c)$ and $\beta(c)$ are continuously differentiable functions of c. The eigenvalues of the Jacobian matrix of system (5) at the origin are $\alpha(c) \pm i\beta(c)$.

THEOREM 9.3.1 **Hopf Bifurcation**

Suppose that $\alpha(0) = 0$, $\alpha'(0) > 0$, and $\beta(0) \neq 0$ and that system (5) is asymptotically stable at the origin for $c = 0$. As c increases through zero, the origin destabilizes and ejects an attracting limit cycle of diameter $K(c)$, where $K(0) = 0$, and $K(c)$ is a continuous and increasing function of c if $c \geq 0$ is sufficiently small. The period of the cycle is approximately $2\pi/|\beta(0)|$ for small $c \geq 0$.

This transition from an asymptotically stable equilibrium point to an unstable equilibrium point enclosed by an attracting limit cycle is a *supercritical Hopf bifurcation*. In the statement of Theorem 9.3.1, the equilibrium point stays fixed at the origin as the parameter c changes, and the Jacobian matrix J of the vector rate function at the origin has the special form given in the margin. In fact, all that is necessary is the following:

☞ The matrix J is $J = \begin{bmatrix} \alpha(c) & \beta(c) \\ -\beta(c) & \alpha(c) \end{bmatrix}$

(1) the system has an equilibrium point for a range of values of c; this point may move along a curve as c changes;

(2) the Jacobian matrix J at the equilibrium point has the complex conjugate eigenvalues $\alpha(c) \pm i\beta(c)$;

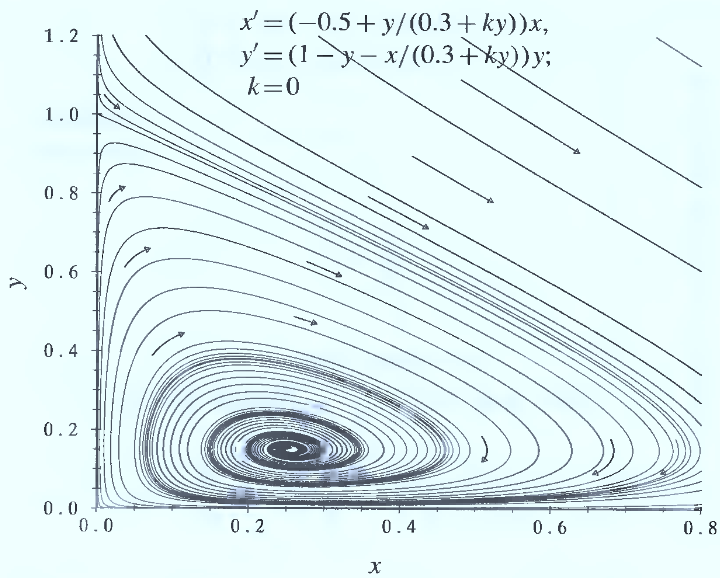


FIGURE 9.3.9 Populations approach an equilibrium point: $k = 0$ (Example 9.3.4).

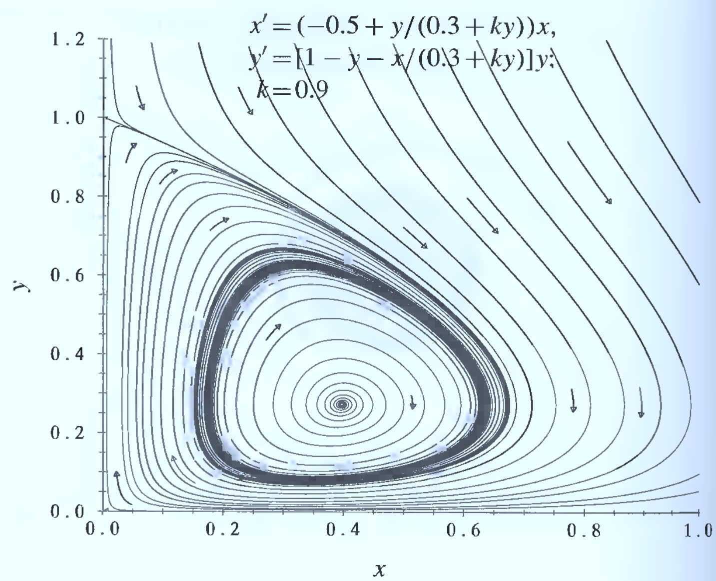


FIGURE 9.3.10 Populations approach a limit cycle after Hopf bifurcation: $k = 0.9$ (Example 9.3.4).

(3) at some value c_0 of the parameter, $\alpha(c_0) = 0$, $\alpha'(c_0) > 0$, and $\beta(c_0) \neq 0$;

(4) the system is asymptotically stable at the equilibrium point if $c = c_0$.

Under these conditions, as c increases through c_0, a bifurcation to an attracting limit cycle must occur.

EXAMPLE 9.3.3

A Hopf Bifurcation

System (2) of Example 9.3.2 is a special case of system (5) with $\alpha(c) = c$, $\beta(c) = 5$, $P = -x(x^2 + y^2)$, and $Q = -y(x^2 + y^2)$. P and Q have order 3 and are twice continuously differentiable. The other conditions, $\alpha(0) = 0$, $\alpha'(0) > 0$, and $\beta(0) \neq 0$, for a Hopf bifurcation are also satisfied. So, the attracting cycle in Figure 9.3.7 is a limit cycle generated by a Hopf bifurcation as c increases through 0.

☞ See Problem 10.

In a *subcritical Hopf bifurcation*, an unstable spiral point stabilizes and spawns a repelling Hopf limit cycle as the parameter c changes (here $\alpha'(0) < 0$). There are other types of Hopf bifurcations in which the equilibrium point moves as c changes. Software can test a system for the presence of a Hopf bifurcation, but we will be satisfied with the visual evidence from numerical solvers. Here is an example.

EXAMPLE 9.3.4

Satiable Predation and Hopf Bifurcation: Figures 9.3.9–9.3.12

When food is plentiful, a predator's appetite is soon satiated, so an increase in the prey population has little effect on the interaction terms in the rate equations. One model for the interactions of a satiable predator population x and a prey population y susceptible to overcrowding is

☞ See Section 7.3.

$$\begin{aligned} x' &= [-a + by/(c + ky)]x \\ y' &= [d - ey - fx/(c + ky)]y \end{aligned} \tag{6}$$

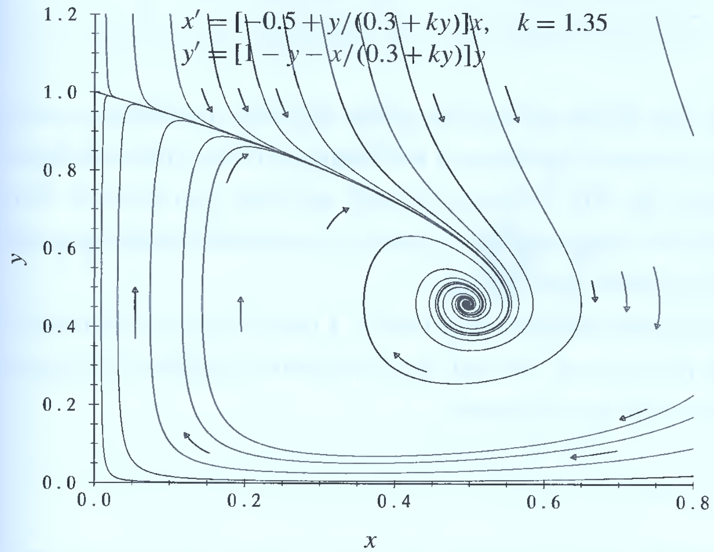


FIGURE 9.3.11 Approach to equilibrium point after reverse Hopf bifurcation: $k = 1.35$ (Example 9.3.4).

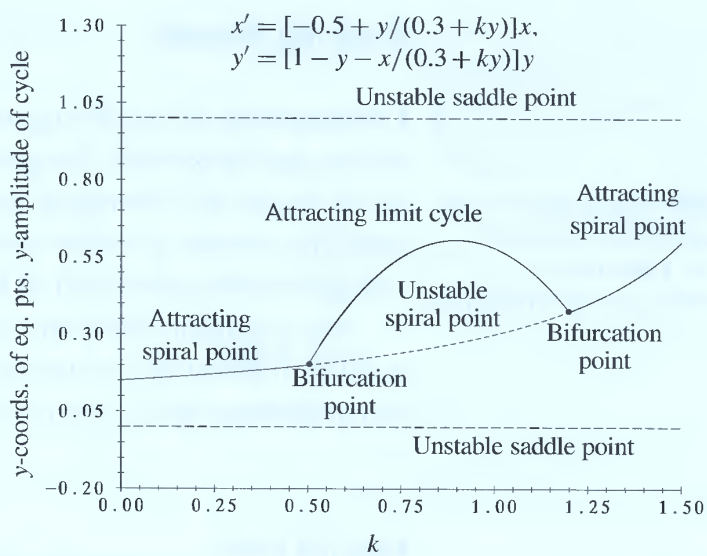


FIGURE 9.3.12 Hopf bifurcation diagram for satiable predation (Example 9.3.4).

where $a, b, \ldots, k$ are positive constants. Hopf bifurcations occur for certain ranges of values of the coefficients. In Figures 9.3.9–9.3.11, the coefficient k is the bifurcation parameter, and the values of the other constants are $a = 0.5$, $b = d = e = f = 1$, and $c = 0.3$. The larger the value of k, the more rapidly the predator's appetite satiates as y increases, so k is known as a satiation coefficient.

Figures 9.3.9–9.3.11 show how the population orbits behave for three values of k: $k = 0$ (no satiation effect), $k = 0.9$, and $k = 1.35$. In the first and third cases, all orbits inside the population quadrant spiral toward an asymptotically stable equilibrium point. In the second case, orbits are pulled toward an attracting limit cycle. As the parameter k continues to increase through 0.5 (approximately) a supercritical Hopf bifurcation takes place. An equilibrium point destabilizes and spawns an attracting limit cycle. As k continues to increase, the cycle's amplitude first increases, but around $k = 0.85$ the amplitude begins to shrink. At $k = 1.2$ the equilibrium point restabilizes and absorbs the cycle in a reverse supercritical Hopf bifurcation. For $k > 1.2$, only an asymptotically stable equilibrium point remains inside the quadrant (Figure 9.3.11).

Figure 9.3.12 shows a bifurcation diagram for system (6), where the values of the constants $a, b, \ldots, f$ are as given in Example 9.3.4. The vertical axis represents the y-coordinate of each of the three equilibrium points of system (6) and the y-amplitude of cycles as measured from the enclosed equilibrium point. The equilibrium points $(0, 0)$ and $(0, 1)$ are always unstable, so we represent them by dashed equilibrium lines. The equilibrium point inside the first quadrant has y-coordinate $0.15/(1 - 0.5k)$. As k increases from zero, this y-coordinate increases. For $0 \leq k \leq 0.5$ and $1.2 \leq k < 2$, the internal equilibrium point is asymptotically stable (solid curve). For $0.5 < k < 1.2$, the point is unstable (dashed curve), and there is an attracting limit cycle (the solid-line hump of the extreme y-value on the cycle).

☞ Since the Jacobian matrices at $(0, 0)$ and $(0, 1)$ have real eigenvalues with opposite signs, each of these points is a saddle point.

Looking Ahead

☞ See Section 9.4 and the WEB SPOTLIGHT ON CHAOS IN A NONLINEAR CIRCUIT.

Contemporary research suggests that bifurcations can often explain mysterious oscillations and behavioral changes in physical systems. Oscillations in the concentrations of chemicals in a chemical reactor, the life cycles of a cell, and the van der Pol limit cycle in a electrical circuit seem to be triggered as a system parameter is changed and a Hopf or some other kind of bifurcation takes place.

The conditions for a bifurcation are often hard to verify. Once we have identified a possible bifurcation parameter in the system, we can use computer graphics to display visual evidence that a bifurcation event has occurred.

PROBLEMS

Saddle-node, Transcritical, Pitchfork Bifurcations. Describe the bifurcations in each system. Find the values of c at which a bifurcation occurs, and identify the bifurcation as of saddle-node, transcritical, or pitchfork type. Saddle-node bifurcations are discussed in the text. As a parameter is changed in a *transcritical bifurcation*, an asymptotically stable node and a saddle point move toward each other, merge, and then emerge with the node now a saddle, and the saddle a node. As a parameter is changed in a *pitchfork bifurcation*, an asymptotically stable equilibrium point suddenly splits into three equilibrium points, two of which are asymptotically stable and the third unstable. Plot graphs of orbits for values of c before, at, and after the bifurcation. Sketch bifurcation diagrams. [*Hint*: use values of c near zero. See Section 2.9.]

1. $x' = c + 10x^2, \; y' = x - 5y$
2. $x' = cx - x^2, \; y' = -2y$
3. $x' = cx + 10x^2, \; y' = x - 2y$
4. $x' = cx - 10x^3, \; y' = -5y$
5. $x' = cx + x^5, \; y' = -y$

Hopf Bifurcation. Show that each system in Problems 6–9 experiences a Hopf bifurcation at $c = 0$. Plot orbits before, at, and after the bifurcation. Draw bifurcation diagrams for Problems 6 and 7 showing the y-amplitude of the limit cycle and the y-coordinate of each equilibrium point as functions of c. [*Hint*: write the ODEs in polar coordinates.]

6. $x' = cx + 2y - x(x^2 + y^2), \; y' = -2x + cy - y(x^2 + y^2)$
7. $x' = cx - 3y - x(x^2 + y^2)^3, \; y' = 3x + cy - y(x^2 + y^2)^3$
8. $x' = y - x^3, \; y' = -x + cy - y^3$
9. *Rayleigh's Equation* The Rayleigh ODE $z'' + c[(z')^2 - 1]z' + z = 0$, which is equivalent to the system $x' = y, \; y' = -x + c(1 - x^2)y$ if we set $z = x, \; z' = y$.

Hopf Bifurcation Investigation.

10. *Subcritical Bifurcation* Show that the origin for $x' = -cx + y + x(x^2 + y^2), \; y' = -x - cy + y(x^2 + y^2)$ bifurcates from an unstable spiral point to a stable spiral point surrounded by a repelling limit cycle as c increases through zero. Plot orbits and a bifurcation diagram.

11. *Hopf Bifurcation: Moving Equilibrium Point* The system

$$x' = c(x - 5c) + (y - 5c) - (x - 5c)[(x - 5c)^2 + (y - 5c)^2]$$

$$y' = -(x - 5c) + c(y - 5c) - (y - 5c)[(x - 5c)^2 + (y - 5c)^2]$$

has the unique equilibrium point $P(5c, 5c)$ for each value of c.

(a) Rewrite the system in terms of u, v coordinates centered at P: $u = x - 5c, \; v = y - 5c$. Then write the new system in polar coordinates based at $u = 0, \; v = 0$.

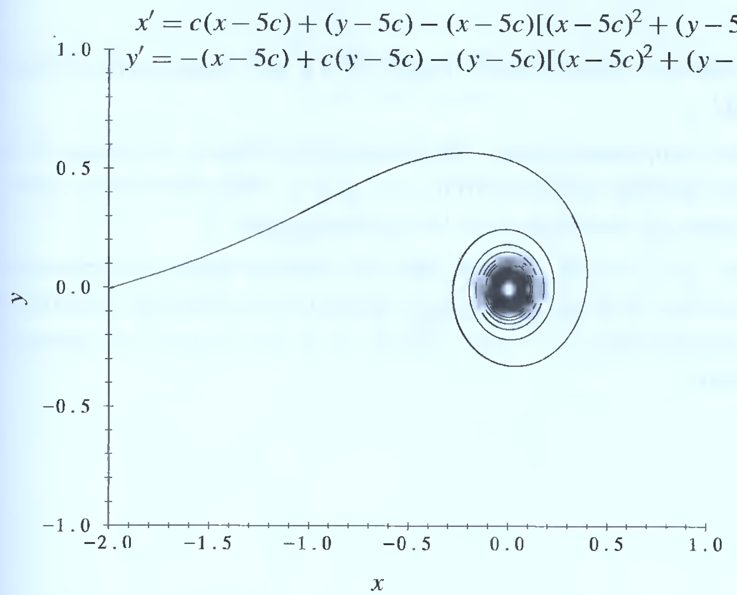


FIGURE 9.3.13 Orbit at the Hopf bifurcation: $c = 0$ [Problem 11**(b)**].

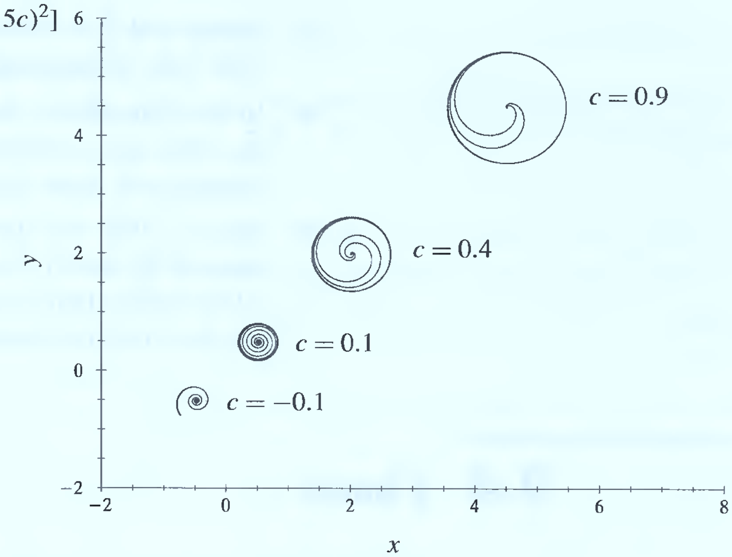


FIGURE 9.3.14 Orbits before and after the Hopf bifurcation [Problem 11**(c)**].

(b) Use **(a)** and show that the system has a supercritical Hopf bifurcation at $c = 0$. Show that for $c > 0$, the circle of radius $\sqrt{c}$ centered at $P(5c, 5c)$ [in x-, y-coordinates] is an orbit. See Figures 9.3.13 and 9.3.14.

(c) Plot orbits for $c = -0.1, 0.1, 0.4, 0.9$ and describe what is happening. [*Hint*: see Figure 9.3.14 where we plot orbits for each of these values of c on a single graph.]

Modeling Problems.

12. *Satiable Predation and Bifurcation* This problem continues the exploration of Example 9.3.4. Let $a = 0.5,\ d = e = f = 1,\ c = 0.3,\ k = 0.9$, and let b be the bifurcation parameter. Explore what happens as b increases from 0.75 to 3. Any Hopf bifurcations?

13. *The Autocatalator and Bifurcation* The autocatalator models a chemical reaction in which the concentration of a precursor W decays exponentially, generating a new species X in the process. Species X decays to Y and at the same time reacts autocatalytically with Y, the latter reaction creating more Y than is consumed. Y decays in turn to Z. The rate equations for the concentrations w, x, y, z are

$$w' = -aw, \quad x' = aw - bx - \alpha xy^2, \quad y' = bx + \alpha xy^2 - cy, \quad z' = cy$$

where the rate coefficients a, b, c, and α are positive parameters and the independent variable is time. In what appears to be a kind of Hopf bifurcation, the concentrations of the chemical intermediates X and Y undergo violent oscillations for certain ranges of the parameters and certain levels of species W. The goal of this project is to understand the behavior of the reaction, given various sets of data and parameters. Address the following points:

☞ See Section 7.1 for more on the autocatalator.

- Treat the system as a nonautonomous planar system by setting $w = w(0)e^{-at}$ and ignoring the rate equation for z. This reduction may help with your solver graphics.
- Why is it reasonable to set $x(0) = y(0) = z(0) = 0$?
- After some initial oscillations in x and y, all four concentrations behave as expected as time increases if $w(0) = 500,\ a = 0.002,\ b = 0.14,\ \alpha = c = 1.0,\ 0 \le t \le 1500$. What levels do the four concentrations approach as t becomes large?
- Repeat with $b = 0.08$ instead of 0.14, and $w(0) = 5000$. What is happening here?