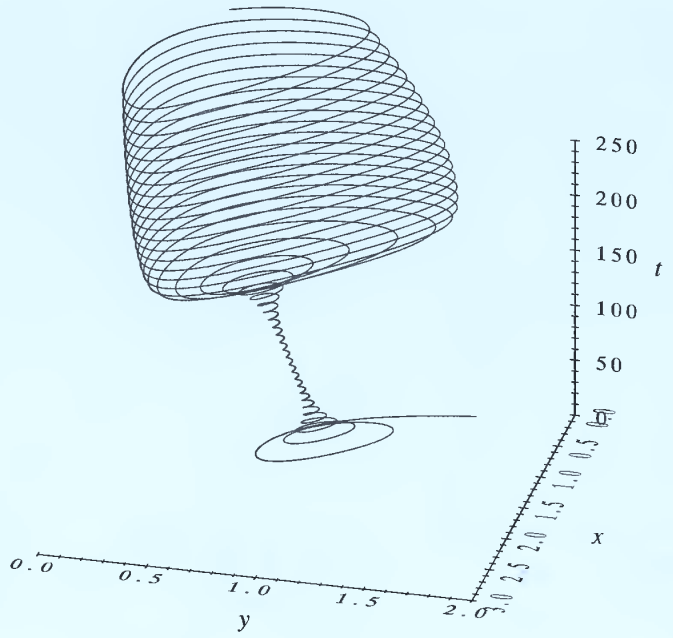
CHAPTER

7



This time-state curve envelops a wineglass.
Take a look at Example 7.1.1.

Nonlinear Differential Systems

In this chapter we focus on natural processes that are modeled by nonlinear differential systems. We use numerical solvers to create displays of solutions of initial value problems and examine the sensitivity of the solution of an IVP to changes in the initial data and the rate functions.

7.1 Chemical Kinetics: The Fundamental Theorem

A chemical reaction is a process in which *reactants* combine, interact, and recombine to form products or other reactants. A *chemical reactor* is a vessel in which the reaction takes place. Chemists want to know how the concentrations of the reactants in the reaction evolve over time. Mathematical models are often as useful as empirical studies for this purpose.

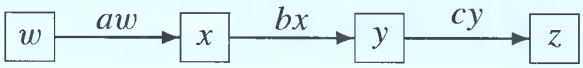
Chemical Reactions: Law of Mass Action

Suppose a reactant in a chemical reaction changes into intermediate reactants, and the intermediates change into a final product. We adapt the compartment model idea and

the Balance Law to model this process. Capital letters denote the various reactants, lowercase letters the corresponding concentrations measured in appropriate chemical units. We assume that the volume of the solution in the reactor remains constant.

In the reaction diagrammed below, reactant W becomes the intermediate X (written as $W \to X$) at the rate of aw concentration units of W per unit of time. Simultaneously, $X \to Y$ at the rate bx, and Y turns into the final product Z at the rate cy. Let's use a compartment model to describe how these concentrations change with time during reactions, writing the reaction rates over the arrows:

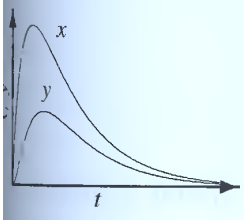
☞ We also discuss compartment models in Chapters 1, 2, Section 6.1, the SPOTLIGHT ON MODELING: COLD MEDICATION I, and the SPOTLIGHT ON MODELING: COLD MEDICATION II.



The constants a, b, and c are positive. Since the transformation rate of each reactant into another is proportional to its concentration, the reactions are first-order. The Balance Law for each reactant gives four linear rate equations in the concentrations:

$$w' = -aw, \qquad x' = aw - bx, \qquad y' = bx - cy, \qquad z' = cy \tag{1}$$

System (1) is a linear cascade. Let's impose the initial conditions $w(0) = w_0$, $x(0) = 0$, $y(0) = 0$, $z(0) = 0$ and describe how to solve the cascade from the top down. In particular, start with the reactant W to obtain $w(t) = w_0 e^{-at}$. Substitute this expression for w into the linear rate equation for x and then, setting $x(0) = 0$, solve the second ODE in (1) for $x(t)$, and so on, step by step down the line. Typical graphs for $x(t)$ and $y(t)$ are sketched in the margin. These intermediate concentrations reach maximum values, and then diminish exponentially (see Problem 18).



In recent times nonlinear autocatalytic reactions have attracted a great deal of attention. In an *autocatalytic reaction* a chemical reactant promotes its own production. Here's an example: two units of Y react with one unit of X to produce three units of Y, a net gain of one unit of Y. In this reaction the rate of transformation from one reactant into another does *not* follow a first-order rate law.[1] In order to model this kind of reaction, we use the following empirically-based principle:

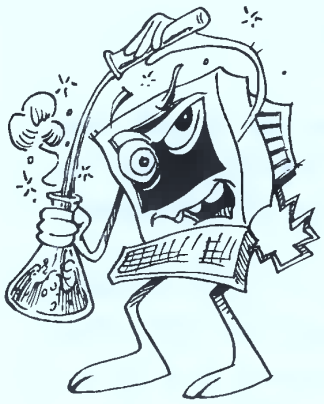
☞ The reactant and product molecules don't need to be distinct.

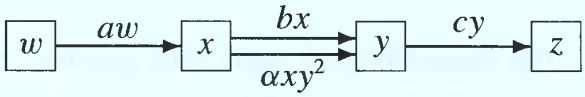
Chemical Law of Mass Action. Suppose that n reactant molecules $X_1, \ldots, X_n$ react to produce m product molecules $P_1, \ldots, P_m$ in one step of a reaction. Then the rate of decrease of the concentration of each reactant molecule and the rate of increase of the concentration of each product molecule is proportional to the product of the concentrations of the n reactants.

[1] In 1951, the Russian chemist Boris Pavlovich Belousov (1894–1970) discovered a specific chemical reaction that behaved like the autocatalator, but no one believed him and his work was ignored. In disgust he abandoned his chemical research, and it was only years later that the importance of his work was acknowledged. Belousov had a turbulent life, starting out as a young revolutionary in the tsarist days. After the Communist Revolution of 1917, he joined the Soviet Army and rose to the rank of Brigade Commander. It was only after retirement from the army that he began his career as a research chemist. A decade after his death his work was recognized with the highest civilian award of the Soviet era. For more on the chemistry and the mathematics of these reactions, see P. Gray and S. K. Scott, *Chemical Oscillations and Instabilities* (Oxford: Clarendon Press, 1990), and S. K. Scott, *Chemical Chaos* (Oxford: Clarendon Press, 1991).

For example, in each of the reactions modeled by the ODEs in system (1), there is one reactant and one product (so $n = 1$, $m = 1$ in the above law).

Apply this principle to the *autocatalytic* reaction $X + 2Y \to 3Y$ which augments the first-order reaction $X \to Y$. The Chemical Law of Mass Action implies that the rate of decrease of X in the autocatalytic step is αxy^2 (α is a positive rate constant), while the rate of increase of Y is $3\alpha xy^2 - 2\alpha xy^2 = \alpha xy^2$. (The rate of decrease of X and the rate of increase of Y are equal by coincidence in this model.) Now let's augment the autocatalytic reaction $X + 2Y \to 3Y$ with two other reactions $W \to X$ and $Y \to Z$ which follow first order rate laws. The augmented compartment model with the relevant rates is





Start with a positive initial concentration of W and zero initial concentrations for the other reactants and the end product to obtain the nonlinear initial value problem:

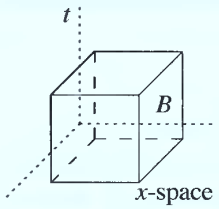
$$\begin{aligned} w' &= -aw, & w(0) &= w_0 \\ x' &= aw - bx - \alpha xy^2, & x(0) &= 0 \\ y' &= bx - cy + \alpha xy^2, & y(0) &= 0 \\ z' &= cy, & z(0) &= 0 \end{aligned} \tag{2}$$

Since the system in IVP (2) is nonlinear there is very little hope of finding a solution formula. However, given numerical values for the data w_0, a, b, c, and α, we can use a numerical solver to see how the four concentrations $w(t)$, $x(t)$, $y(t)$, and $z(t)$ behave. We do just that in Example 7.1.1 and the chapter opening figure.

States, Systems, and Solutions

Many processes lead to *first-order systems* of ODEs in *normal form*

$$\begin{aligned} dx_1/dt &= f_1(t, x_1, x_2, \ldots, x_n) \\ dx_2/dt &= f_2(t, x_1, x_2, \ldots, x_n) \\ &\vdots \\ dx_n/dt &= f_n(t, x_1, x_2, \ldots, x_n) \end{aligned} \tag{3}$$



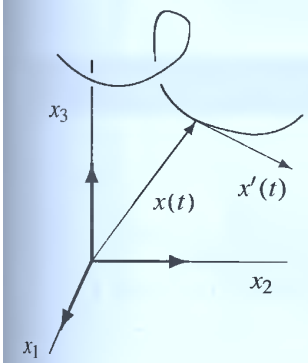
☞ Boxes include their boundaries.

in the *state variables* $x_1, \ldots, x_n$ and the independent variable t, representing time. The *rate functions* f_i are defined on some common box B of $tx_1x_2 \cdots x_n$-space. A box in $\mathbb{R}^{n+1}$ generalizes an interval in $\mathbb{R}^1$ and a rectangle in $\mathbb{R}^2$.

Including initial conditions, we get the *initial value problem*

$$\begin{aligned} x_1' &= f_1(t, x_1, x_2, \ldots, x_n), & x_1(t_0) &= a_1 \\ x_2' &= f_2(t, x_1, x_2, \ldots, x_n), & x_2(t_0) &= a_2 \\ &\vdots \\ x_n' &= f_n(t, x_1, x_2, \ldots, x_n), & x_n(t_0) &= a_n \end{aligned} \tag{4}$$

where the initial data point $(t_0, a_1, \ldots, a_n)$ lies in the box B. The Fundamental Theorem at the end of this section states that IVP (4) has a unique solution $x_1 = x_1(t)$, $x_2 = x_2(t), \cdots, x_n = x_n(t)$ if the rate functions are continuously differentiable in B.



Geometry of Solutions of Systems

We introduce a compact notation for system (3) by working first with three state variables. Locate each point in the state space $\mathbb{R}^3$ by its *position vector* $x = [x_1 \;\; x_2 \;\; x_3]^T$ where x_1, x_2, and x_3 are the cartesian coordinates of the point. If the coordinates are functions of t, the derivative of $x(t)$ is the vector $x'(t) = [x_1'(t) \;\; x_2'(t) \;\; x_3'(t)]^T$. The vector $x'(t)$ is tangent to the curve traced out by the position vector $x(t)$. See the figure in the margin. In other words,

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \quad \text{and} \quad x'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix}$$

The discussion above generalizes to n state variables. Denote the *state vector* x, the *initial state vector* x^0, and the *rate function vector* f by the column vectors

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \qquad x(t_0) = x^0 = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}, \qquad f = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$$

and define the derivative x' entrywise. Using this compact notation, IVP (4) becomes

$$x' = f(t, x), \qquad x(t_0) = x^0 \tag{5}$$

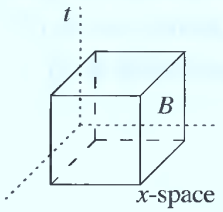
This is exactly what we did in Chapter 6 for linear systems (but using a system matrix to express the rate function vector).

A function vector $x(t)$ defined on some t-interval I which contains the initial point t_0 is a *solution* of IVP (4) if $x'(t) = f(t, x(t))$ for all t in I and $x(t_0) = x^0$. Sometimes there are formulas for solutions of systems (as we saw in Chapter 6 for linear systems with constant coefficients), but for a system which models an intricate natural process, that is rarely true. So the focus we take in this chapter is on numerical solutions and qualitative properties of solutions, not on solution formulas.

A solution of a system of first-order ODEs determines curves whose behavior highlights properties of the solution.

❖ **Time-State Curves, Component Curves, Orbits.** Suppose that $x = x(t)$ is a solution of the system $x' = f(t, x)$. The point $(t, x(t))$ traces out a *time-state curve* in the time-state space $\mathbb{R}^{n+1}$ of the variables $t, x_1, x_2, \ldots, x_n$. The projection of a time-state curve onto the tx_j-plane is the x_j-*component curve*. The projection of a time-state curve onto the $x_1x_2 \cdots x_n$ state space is an *orbit*. A collection of orbits is an *orbital portrait*.

The Fundamental Theorem for Systems



Theorem 7.1.1 guarantees that, under mild conditions on the rate function f, IVP (5) has exactly one maximally extended solution and that solution is a continuous function of the data x^0 and f.

THEOREM 7.1.1

Fundamental Theorem for Systems

Consider the IVP for n state variables

$$x' = f(t, x), \qquad x(t_0) = x^0$$

where all the functions f_i and $\partial f_i/\partial x_j$ are continuous on a box B in $(n+1)$-dimensional tx-space, and (t_0, x^0) is a point of B.

Existence: The IVP has a solution on some t-interval containing t_0.

Uniqueness: The IVP has at most one solution on any t-interval containing t_0.

Extension: We can extend the solution to a t-interval containing t_0 for which the time-state curve lies in B and extends to the boundary of B as t tends to either endpoint of the interval.

Continuity in the Data: The solution is continuous in the data x^0 and f.

☞ For continuity in the data for first-order ODEs see the SPOTLIGHT ON CONTINUITY IN THE DATA in Chapter 2.

From now on, we only consider systems for which the Fundamental Theorem 7.1.1 applies. We always assume that the solution of an IVP and the corresponding time-state curve are maximally extended. The Fundamental Theorem implies that a maximally extended time-state curve can't suddenly die inside a box B where the conditions on f hold, so the curve must cross B from boundary to boundary. Another consequence of the Fundamental Theorem is that no two time-state curves can touch inside a box where the hypotheses on f hold, but this is *not* generally true for component curves and orbits.

As an illustration of the Fundamental Theorem, let's look at a nonautonomous system for the concentrations of the chemical intermediates in the autocatalytic reaction modeled by system (1).

EXAMPLE 7.1.1

☞ See the WEB SPOTLIGHT ON SCALING AND UNITS for the details of the scaling process.

Component Curves, Orbit, Time-State Curve for a Nonlinear IVP

We scale the variables, coefficients, and initial data for the model system (2) of an autocatalytic reaction to get dimensionless quantities t, w, w_0, x, y, z, a, b, c, and α. We use the same names for the dimensionless variables as in (2), and give to the dimensionless constants the values used by P. Gray and S.K. Scott (see footnote 1 on page 445):

$$w_0 = 500,\ a = 0.002,\ b = 0.08,\ c = 1, \alpha = 1 \tag{6}$$

From the ODE for $w(t)$ and the value of w_0 in (6), we see that $w(t) = 500e^{-0.002t}$, so

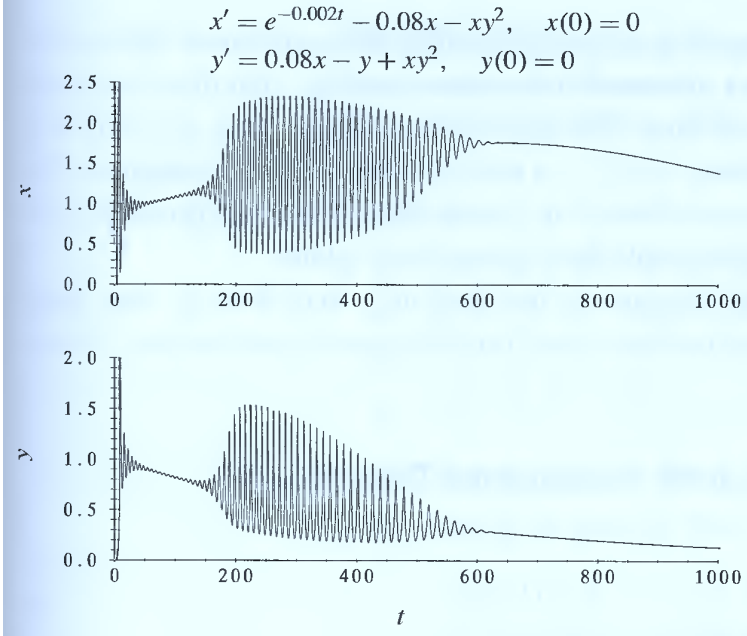


FIGURE 7.1.1 Autocatalytic oscillations turn on and then switch off (Example 7.1.1).

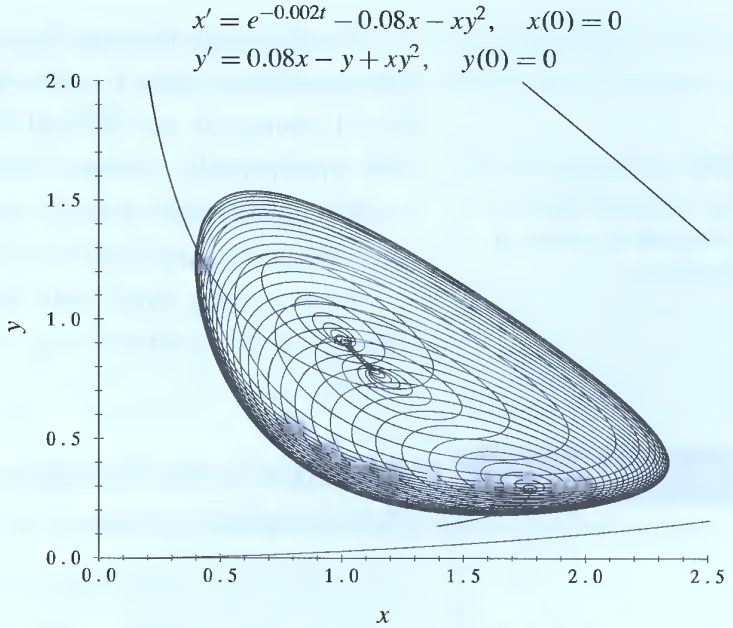


FIGURE 7.1.2 The orbit of the autocatalytic interaction between the intermediates (Example 7.1.1).

the second and third IVPs in (2) decouple to become

$$\begin{aligned} x' &= f_1(t, x, y) = e^{-0.002t} - 0.08x - xy^2, & x(0) &= 0 \\ y' &= f_2(t, x, y) = 0.08x - y + xy^2 & y(0) &= 0 \end{aligned} \tag{7}$$

The rate functions f_1 and f_2 are continuous functions for all t, x and y. In addition, the first-order partial derivatives with respect to x and y

$$\begin{aligned} \partial f_1/\partial x &= -0.08 - y^2, & \partial f_1/\partial y &= -2xy \\ \partial f_2/\partial x &= 0.08 + y^2, & \partial f_2/\partial y &= -1 + 2xy \end{aligned}$$

are continuous for all t, x, and y. So a maximally extended time-state curve of IVP (7) reaches backward and forward across every box it intersects in txy-space.

Figure 7.1.1 shows unusual and unexpected oscillations in the concentrations in the intermediates. These oscillations are internally generated and are *not* due to oscillations in external factors such as temperature or pressure. After some initial swings the concentrations behave almost normally until around $t = 170$ when the violent oscillations in the concentrations begin. The oscillations stop around $t = 600$ and the concentrations of the intermediates gradually decline.

The corresponding orbit (Figure 7.1.2) self-intersects, but this does not contradict the Fundamental Theorem because the first rate function in IVP (7) depends explicitly on t. The orbit transits each point of self-intersection at different times. The appearance of the orbital arc in Figure 7.1.2 when it emerges from the tangle of the oscillations suggests that the orbit is heading back to the origin as the reaction nears completion. The chapter opening figure shows the time-state curve of IVP (7) over the time interval $7 \le t \le 377$.

Even though the rate functions of a differential system are continuous and continuously differentiable for all values of time and the state variables, that does *not* mean that all solutions are defined for all time. The first-order example, $y' = y^2$, $y(0) = 1$, with a maximally extended solution $y = 1/(1-t)$, $t < 1$, shows what can happen. The solution approaches infinity as time increases to 1 even though the rate function y^2 is continuous and continuously differentiable throughout the ty-plane.

☞ In Example 2.7.1 we called this behavior "escaping to infinity in finite time."

Over a long time span small changes in the data may may lead to very large changes in the solution (e.g., see the WEB SPOTLIGHT ON PENDULUM MOTION: SENSITIVITY).

EXAMPLE 7.1.2

Illustration of the Conclusions of the Fundamental Theorem

The uncoupled system

$$\begin{aligned} x_1' &= x_1^2, \quad & x_1(t_0) &= b_1 \\ x_2' &= 2x_2, \quad & x_2(t_0) &= b_2 \end{aligned} \tag{8}$$

meets the conditions of the Fundamental Theorem in all of $\mathbb{R}^3$. The unique solution of system (8) may be obtained by separating variables in each rate equation and solving. For example, the solution formula for x_2 is

$$x_2(t) = b_2 e^{2(t-t_0)}, \quad -\infty < t < \infty \tag{9}$$

Solving the rate equation for x_1 is more involved. First note that $x_1(t) = 0$ for all t is the solution corresponding to $b_1 = 0$. If $b_1 \neq 0$, the variables may be separated. After integrating, using the initial data, and solving for x_1 in terms of t,

$$x_1(t) = \frac{b_1}{1 - b_1(t - t_0)}, \quad \text{where} \begin{cases} t < t_0 + 1/b_1 & \text{if } b_1 > 0 \\ t > t_0 + 1/b_1 & \text{if } b_1 < 0 \end{cases} \tag{10}$$

Formula (10) also gives the solutions $x_1(t) = 0$, $-\infty < t < \infty$, if $b_1 = 0$. The change in the t-interval, depending on the sign of b_1, is dictated by the need to keep t_0 inside the interval where the solution is defined. In either case, the solution has *finite escape time* since $x_1(t)$ becomes unbounded as t tends to the finite value $t_0 + 1/b_1$. This restriction on the t-interval of the definition of $x_1(t)$ forces the same restriction on the interval of definition of $x_2(t)$ since the full solution of IVP (8) requires both $x_1(t)$ and $x_2(t)$ to be defined on a common interval. For each value of t_0, b_1, and b_2 IVP (8) has a unique solution, although the solution formula and the t-interval depend on the values of t_0 and b_1. Moreover, the solution given by formulas (9) and (10) is a continuous function of t, t_0, b_1 and b_2 because quotients of continuous functions are also continuous (except where the denominator is zero).

Looking Back

According to the Fundamental Theorem 7.1.1, the solution $x(t)$ of the IVP $x' = f(t, x)$, $x(t_0) = x^0$, is continuous in the data f and x^0 if the components of f and their first-order partial derivatives are continuous. So small changes in $x(t)$ can be guaranteed (at least over a sufficiently small time span containing t_0) by putting bounds on the allowable changes in f and x^0.

PROBLEMS

The Fundamental Theorem. For Problems 1–7 verify that each IVP satisfies the hypotheses of the Fundamental Theorem. Then solve the IVP.

1. $x_1' = x_2, \quad x_2' = -x_1 - 2x_2; \quad x_1(0) = 1, \quad x_2(0) = 1$ [*Hint*: note $x_1'' + 2x_1' + x_1 = 0$.]
2. $x_1' = x_2, \quad x_2' = -x_1 - 2x_2; \quad x_1(0) = a, \quad x_2(0) = b$
3. $x_1' = 2x_1 + e^{2t}, \quad x_2' = -4x_2; \quad x_1(0) = a, \quad x_2(0) = b$
4. $x_1' = x_2, \quad x_2' = -9x_1; \quad x_1(0) = a, \quad x_2(0) = b$
5. $x_1' = x_2^3, \quad x_2' = -x_1^3; \quad x_1(0) = a, \quad x_2(0) = b$
6. $x_1' = -x_1^3, \quad x_2' = -x_2 \sin t; \quad x_1(0) = 1, \quad x_2(0) = 1$
7. $x_1' = x_2, \quad x_2' = -26x_1 - 2x_2, \quad x_3' = x_3/2; \quad x_1(0) = x_2(0) = x_3(0) = 1$
8. *Here Is the Solution; What Is the System?* Alex Trebek hands you two function vectors

$$u = [\sin t \;\; t]^T, \qquad v = \left[te^t \;\; t^3\right]^T$$

and asks you for continuously differentiable functions $f(x, y, t)$ and $g(x, y, t)$ for which u and v solve the system of ODEs $x' = f$, $y' = g$. Can you do it? Explain why, or why not.

Orbits. For each IVP in the indicated problem plot the nine orbits corresponding to all possible combinations of $a, b = -1, 0, 2$. Identify any orbits that are equilibrium points or cycles.

9. Problem 1
10. Problem 2
11. Problem 3
12. Problem 4
13. Problem 5
14. Problem 6

Chemical Rate Equations. Given the diagrams below for the chemical reaction steps, write out the ODEs for the concentrations. The quantities k_1, k_2, k are positive rate constants. [*Hint*: the rate equation for the concentration x of reactant X in Problem 15 is $x' = -k_1 xy$.]

15. $X + Y \xrightarrow{k_1} Z \xrightarrow{k_2} W$
16. $X + 2Y \xrightarrow{k} Z$
17. $X + 2Y \xrightarrow{k} 6Y + W$

The Autocatalator. The rate constant α in the autocatalator system (2) "turns on" the autocatalytic reaction if its value is positive. The reaction "turns off" if $\alpha = 0$ (see system (1)).

18. Solve the cascade in system (1) with the initial conditions. $w(0) = w_0 > 0$, $x(0) = y(0) = z(0) = 0$.
19. *Comparison of Linear Model ODE with Autocatalator* Compare the w, x, y, and z-component curves of the nonlinear autocatalytic system (2) where $\alpha = 0.002$ with those of the linear system (1) as follows: Use the data in (6) and plot each component curve over the time span $0 \le t \le 1000$. Overlay the $\alpha = 0$ and $\alpha = 0.002$ curves for each of the four reactants. How much does the autocatalytic reaction affect the long-term behavior of the various concentrations? The short-term behavior? Explain what you see.
20. *Sensitivity to Changes in Data: Chemical Intermediates* Replace the coefficient 0.002 in IVP (7) by α and let α have the value 0.002, then 0.02, and finally 0.2. Plot the tx- and ty-component curves for $0 \le t \le 10$, then for $0 \le t \le 1000$. Are the curves sensitive to the changes in α over the short time span? Over the long time span? Explain.

Changing the Data. The two-dimensional autocatalator orbit of Figure 7.1.2 is the projection of the time-state curve of IVP (7) onto the xy-plane. Problems 21–25 extend IVP (7) by varying the initial concentration $w(0) = w_0$ of the reactant w. [*Hint*: see Multimedia Module 8 in ODE Architect; also see the Library entry The Autocatalator Reaction under Chemical Models.]

21. Solve system (2), using the data in (6), but with $w_0 = 50$, then 100. What unusual features do you see in the x- and y-component graphs in comparison to what is visible in Figure 7.1.1?
22. Find the value of w_0 for which you first see sustained oscillations in the component graphs.
23. Plot time-state curves like the chapter opening figure, for $w_0 = 50, 250, 500, 800$.

24. Describe the behavior of solutions for various values of w_0 over the range $100 \le w_0 \le 500$.

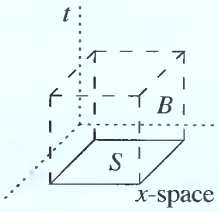
25. *Two-Dimensional Autocatalator, Turning Oscillations On and Off* It may be possible to turn the x and y oscillations on or off by changing any one of the five parameters w_0, a, b, c, and α. That is the aim of the project: change a parameter up or down from the value given until the oscillations of the concentrations of the intermediates X and Y disappear. Then explain why you think that happens. Some suggestions:

- Duplicate Figures 7.1.1 and 7.1.2.
- Duplicate the wineglass in the opening figure of the chapter. What happens if you plot the solution curve for $0 \le t \le 2000$ rather than the truncated curve of the wineglass, $7 \le t \le 377$?
- Vary the coefficient b from 0.08 up to 0.14, keeping all other parameters fixed. What happens? Any explanation?
- Vary the coefficients a from 0.002 and c from 1, but keep the other parameters fixed at the values given. Can you turn off the oscillations?

7.2 Properties of Autonomous Systems, Direction Fields

A differential system is *autonomous* if its rate functions do not explicitly depend on time. Autonomous systems allow us to use special techniques that give valuable information about solution behavior. Let's look at the autonomous IVP

$$x' = f(x), \qquad x(t_0) = x^0 \tag{1}$$



where the conditions of the Fundamental Theorem hold in a box B of tx-space (see the margin figure). Since f doesn't depend on t, we use a box S in x-space, the *state space* of the system, instead of the box B in tx-space. Think of S as the projection of the box B into the x-space along the t-axis. Suppose that x^0 is in S, and t_0 is any real number. We now interpret the four conclusions of Fundamental Theorem 7.1.1 in terms of S instead of B. There are some differences: for example, even if the box S is bounded, the orbit of the unique maximally extended solution curve of IVP (1) need not meet the boundary of S. Indeed, the orbit may stay inside S as $t \to +\infty$ (or as $t \to -\infty$), as we saw in Section 6.5 for some planar linear systems.

Orbits, Equilibrium Points, Cycles of Autonomous Systems

☞ See Sections 2.8 and 3.8 for these properties in the context of first- and second-order autonomous ODEs.

Suppose that $x = x(t)$, $a < t < b$, is a solution of the autonomous system, $x' = f(x)$. Then, for any constant c, the function $x = x(t + c)$, $a - c < t < b - c$, is also a solution. The two solutions determine exactly the same orbit in state space because the two time-state curves have the same projection onto the state space. So, it doesn't matter when the clock is started to generate an orbital arc of an autonomous system; only the total time span $b - a$ matters.

Next, we show that distinct orbits of an autonomous system never meet.

THEOREM 7.2.1 Separation of Orbits

Suppose that the autonomous system is $x' = f(x)$ and that the functions f_i and $\partial f_i/\partial x_j$, $i, j = 1, \ldots, n$, are continuous on a box S in state space. Then orbits of two different maximally extended solutions never meet in S.

Proof. Suppose that two orbits *do* meet at a point. Since the rate functions don't depend on time, reset the clock so that both orbits are at the point at the same time. By the uniqueness property for IVPs, the orbits coincide. ■

If a solution $x(t)$ stays constant for all time, it is an *equilibrium solution*. The corresponding time-state curve is a straight line in tx-space parallel to the t-axis. The orbit is a point in state space (an *equilibrium point*). No nonconstant orbit ever touches an equilibrium point, because to do so would violate the Separation of Orbits Theorem 7.2.1.

Equilibrium points correspond to the zeros of the rate function $f(x)$. There are computational techniques for finding these zeros, but we mostly determine them by inspection. Here's an example.

EXAMPLE 7.2.1 Equilibrium Points for a Planar System

The equilibrium points for the planar autonomous system

$$\begin{aligned} x' &= x - y + x^2 - xy \\ y' &= -y + x^2 \end{aligned} \tag{2}$$

are the points (x, y) in the xy-plane at which both rate functions are zero:

$$x - y + x^2 - xy = 0, \quad \text{and} \quad -y + x^2 = 0 \tag{3}$$

Since $y = x^2$, replace y in the first equation of (3) by x^2:

$$x - x^2 + x^2 - x^3 = 0, \quad \text{or} \quad x(1 - x^2) = 0$$

So, the x-coordinate of an equilibrium point for system (2) must be 0, $+1$, or -1. The second equation in (3) now yields the three equilibrium points $(0, 0)$, $(1, 1)$, $(-1, 1)$.

If an orbit of an autonomous system intersects itself after the passage of T units of time, then restarting the clock at the intersection point guarantees that the same path is traversed again in T units of time, and so on ad infinitum. Therefore, a closed curve orbit arises from a periodic solution.

❖ **Cycles**. Periodic solutions of an autonomous system generate closed orbits, or *cycles*, in state space.

Cycles have the following properties:

THEOREM 7.2.2 Properties of Cycles

Suppose that the autonomous system is $x' = f(x)$ and that the functions f_i and $\partial f_i/\partial x_j$, $i, j = 1, \ldots, n$, are continuous in a box S in state space. Then

- **Cycle Property 1:** no other orbit can touch a cycle in S.
- **Cycle Property 2:** a nonconstant orbit in S that meets itself is a cycle.

Proof. Cycle Property 1 follows from the Separation of Orbits Theorem 7.2.1. Cycle Property 2 holds because at the meeting point, we can reset the clock and regenerate the same closed curve over the same time span. ■

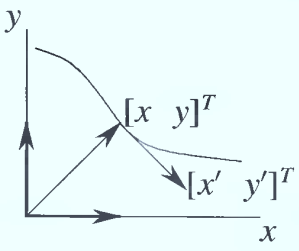
We find cycles by inspecting the system, by constructing solution formulas, and sometimes by more intricate theoretical means. Frequently, however, the main clues that periodic behavior exists show up on the computer screen in the graphs of component curves and orbits. Let's go to the two dimensional state space of a pair of autonomous ODEs, where we can provide visual examples of these concepts.

Planar Autonomous Systems, Direction Fields

For the rest of this chapter, we mostly look at planar autonomous systems

$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \end{aligned} \tag{4}$$

where the real-valued rate functions f and g and their first-order partial derivatives are continuous on a box S in xy-space. We know that precisely one maximally extended orbit passes through each point (x_0, y_0) in S.



The state space of system (4) is the xy-plane. An orbit is a curve described by the endpoint of the position vector from the origin $[x(t) \;\; y(t)]^T$, where $x(t)$, $y(t)$ is a solution of the system (4). As the endpoint of the position vector traces out the orbit in time, the velocity vector $\mathbf{v}(t) = [x'(t) \;\; y'(t)]^T$ of the endpoint is tangent to the orbit (see the figure in the margin). The tangent vector $\mathbf{v}$ at the point (x, y) is $[f(x, y) \;\; g(x, y)]^T$. Since the time t that the orbit arrives at (x, y) is irrelevant, think of $\mathbf{v}$ as a function of x and y, and not of t. The length $\|\mathbf{v}(x, y)\|$ of $\mathbf{v}$ gives the *speed* at which a point traverses the orbit as it passes through (x, y).

Here is another device for visualizing orbital behavior. Place a grid of points on the rectangle $a \le x \le b$, $c \le y \le d$. Draw a tiny line segment centered at each grid point (x, y) and parallel to the vector $\mathbf{v} = f(x, y)\mathbf{i} + g(x, y)\mathbf{j}$. Then scale the segments so that they all have the same length, and no two segments intersect. The resulting field of line segments is a *direction field*.[2]

☞ We introduced direction fields in Section 3.3 for a special kind of planar autonomous system.

[2]The slope fields in Section 2.4 for first-order scalar ODEs $y' = f(t, y)$ lie in the ty-plane and should not be confused with the xy-direction fields for the orbits of planar autonomous systems.

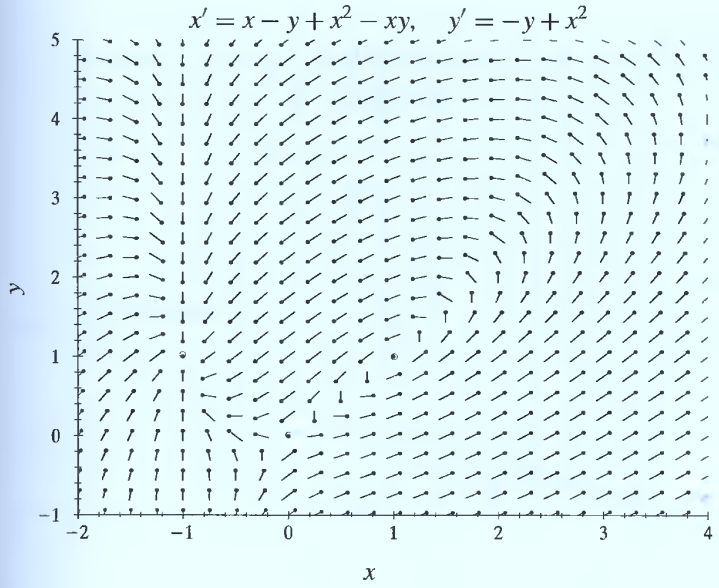


FIGURE 7.2.1 A direction field and three equilibrium points (Example 7.2.2).

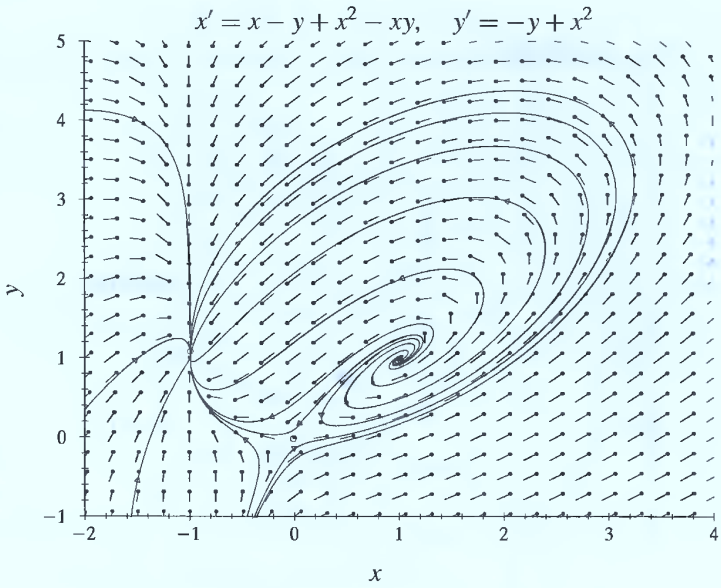


FIGURE 7.2.2 See how the orbits fit the direction field of Figure 7.2.1 (Example 7.2.2).

A continuously differentiable curve in the xy-plane is an orbit of system (4) if the tangent vector at any point (x, y) on the curve is parallel to the *velocity field* vector $f(x, y)\mathbf{i} + g(x, y)\mathbf{j}$. So, a smooth curve in the xy-plane is an orbit of system (4) if it fits a direction field. The direction field line segment centered at the grid point (x, y) is often oriented with an arrowhead that points in the direction of the velocity vector $\mathbf{v}(x, y)$ to show the direction in which orbits are traced out as t increases; we use dots instead of arrowheads.

☞ A sneaky way to visualize orbits. Useful, too!

EXAMPLE 7.2.2

An Unusual Direction Field and Some Orbits

Figure 7.2.1 shows a direction field for the system in Example 7.2.1:

$$x' = x - y + x^2 - xy, \qquad y' = -y + x^2$$

The direction field suggests the nature of the orbits. This view is especially interesting near the system's three equilibrium points: $(0, 0)$, $(1, 1)$, and $(-1, 1)$. Figure 7.2.2 shows orbits that approach $(0, 0)$, but then they seem to veer off. Orbits that head toward $(-1, 1)$ don't turn away, but they can't reach that equilibrium point in finite time. Orbits seem to emerge (as t increases from $-\infty$) from the equilibrium point $(1, 1)$ in a spiral, counterclockwise motion and then head off toward the equilibrium point $(-1, 1)$. Figure 7.2.2 shows how closely orbits fit the direction field.

Let's now take a look at special curves in the state plane that help us understand just where orbits of a planar autonomous system rise, fall, and change direction.

Nullclines

The curves in the xy-plane defined by $f(x, y) = 0$ are the *x-nullclines* for system (4). The curves defined by $g(x, y) = 0$ are the *y-nullclines*. The x-nullclines meet the y-

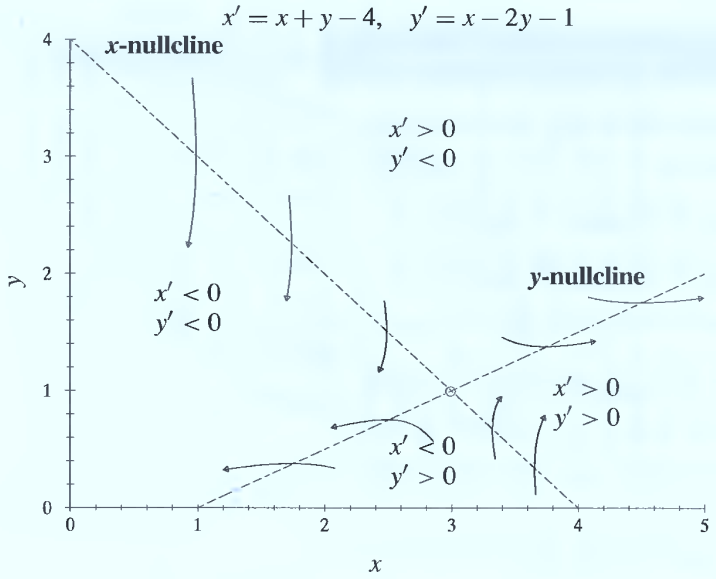


FIGURE 7.2.3 Nullclines, time-directed orbital arcs crossing nullclines (Example 7.2.3).

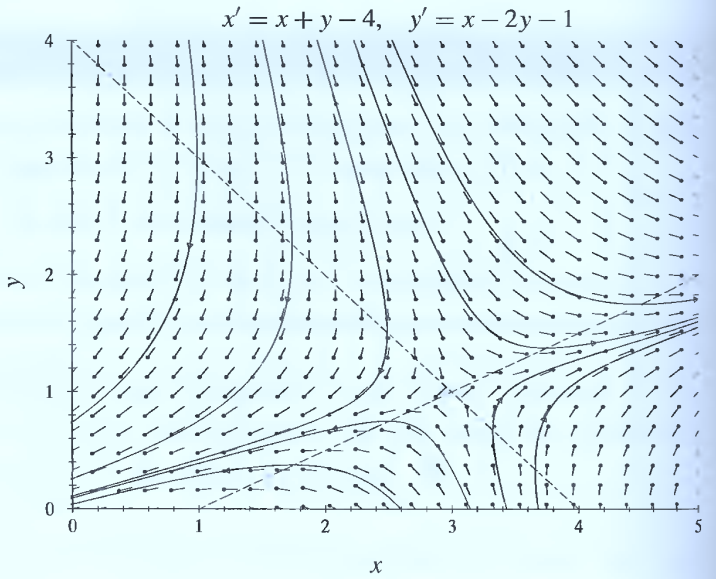


FIGURE 7.2.4 Orbits, direction field, nullclines (Example 7.2.3).

nullclines at the equilibrium points, so sketching the nullclines gives us a way to locate the equilibrium points of system (4). The rate function $f(x, y)$ has a fixed sign on each side of an x-nullcline. Similarly, $g(x, y)$ has a fixed sign on each side of a y-nullcline. The nullclines divide the xy-plane into regions where orbits move to the right ($f > 0$), move to the left ($f < 0$), rise ($g > 0$), or fall ($g < 0$). Knowing where these regions are and the signs of f and g inside them helps us visualize the portrait of the orbits before the actual orbits are even constructed.

☞ This process is like the *sign analysis* of Section 2.8.

EXAMPLE 7.2.3 **Nullclines of a Planar Autonomous Linear System**

The nullclines of the linear system

$$\begin{aligned} x' &= x + y - 4 \\ y' &= x - 2y - 1 \end{aligned} \tag{5}$$

are the straight lines

$$x\text{-nullcline:} \quad x + y - 4 = 0, \qquad y\text{-nullcline:} \quad x - 2y - 1 = 0$$

The two lines cross at the equilibrium point $(3, 1)$ in the xy-plane. The dashed lines in Figure 7.2.3 are the nullclines that give a good indication of orbital behavior. Label each sector formed by x- and y-nullclines according to whether x' and y' are positive or negative in the sector. For example, $x' = x + y - 4$ is positive above the x-nullcline (just check the sign at one point in the sector), so sign analysis tells us that an orbit must move to the right as t increases. Above the y-nullcline the derivative $y' = x - 2y - 1$ is negative, so above both nullclines orbits move to the right ($x' > 0$) and downward ($y' < 0$). A similar sign analysis applies to the other sectors.

With all this information about the signs of x' and y' at hand, sketch arcs of orbits as they cross the nullclines. For example, the arc at the upper left of Figure 7.2.3 starts in the upper sector near the x-nullcline, moves down and to the right, crosses the x-

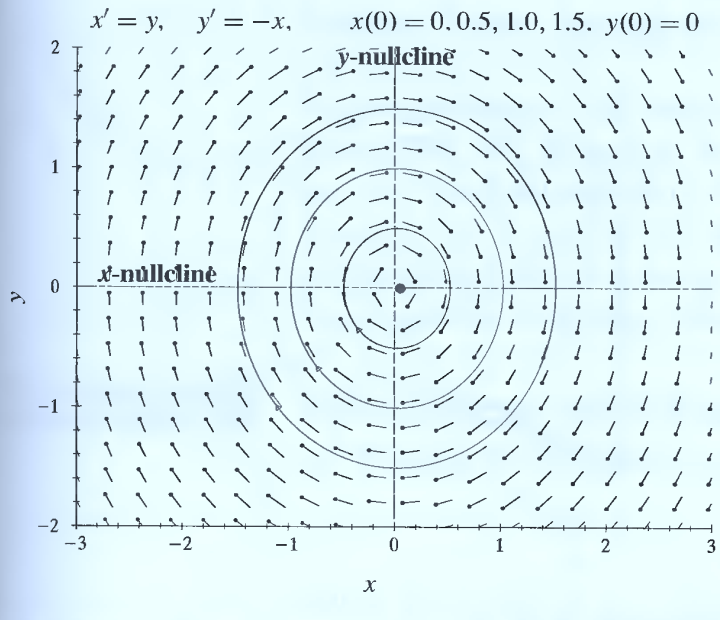


FIGURE 7.2.5 Direction field, nullclines (the axes), equilibrium point (origin), cycles for a harmonic oscillator (Example 7.2.4).

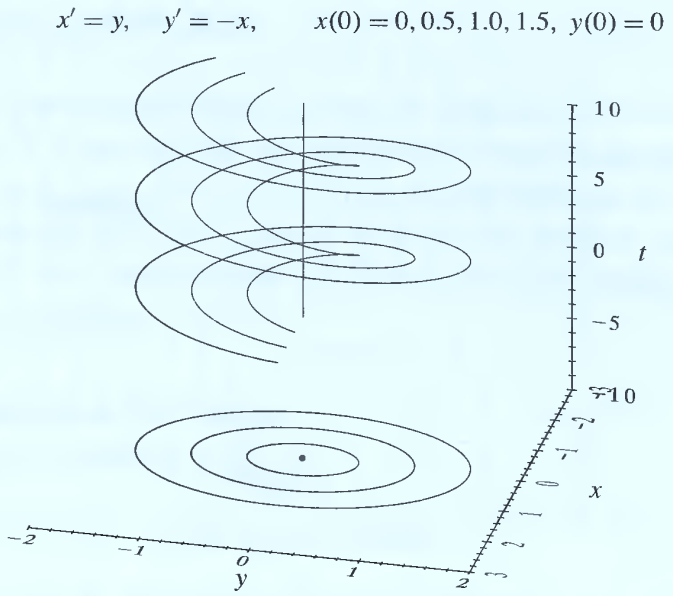


FIGURE 7.2.6 Time-state curves in xyt-space for the harmonic oscillator and the corresponding planar cycles (Example 7.2.4).

nullcline vertically (since $x' = 0$ on the nullcline), and then turns to the left since in this new sector $x' < 0$. The arrowhead shows the direction of time's increase on each orbital arc. See Figure 7.2.4 for some computer-generated orbits of system (5); the orbits do indeed fit the direction field.

Now let's take a system we saw in Section 3.4 as a second-order ODE and find its nullclines, direction field, orbits, and time-state curves.

EXAMPLE 7.2.4

The Harmonic Oscillator

The harmonic oscillator system of Section 3.4 is

$$x' = y, \qquad y' = -\omega^2 x \tag{6}$$

☞ See Problem 1.

where ω is a nonzero constant. The nullclines are

$$x\text{-nullcline: } y = 0, \qquad y\text{-nullcline: } x = 0$$

and $(0, 0)$ is the single equilibrium point. The first-order system (6) is equivalent to the second-order ODE $x'' + \omega^2 x = 0$. The nonconstant solutions of $x'' + \omega^2 x = 0$ are all periodic with a common period $2\pi/\omega$, and the orbits are cycles.

Figure 7.2.5 shows the nullclines crossing at the equilibrium point at the origin. It also shows a direction field and three cycles in the case $\omega = 1$. Figure 7.2.6 illustrates the time-state curves with the orbits below. The constant time-state curve $x = 0$, $y = 0$, corresponding to the equilibrium point at the origin in Figure 7.2.5, shows up as a straight line parallel to the t-axis in Figure 7.2.6. We plot the orbits in Figure 7.2.6 in the plane $t = -10$ so that orbits and time-state curves don't appear to meet.

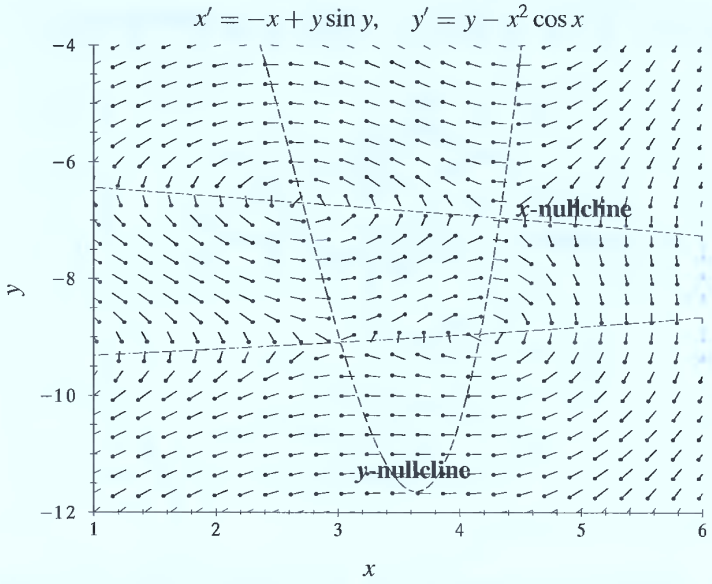


FIGURE 7.2.7 Direction field, nullclines (dashed), and four equilibrium points. The y-nullcline opens upward (Example 7.2.5).

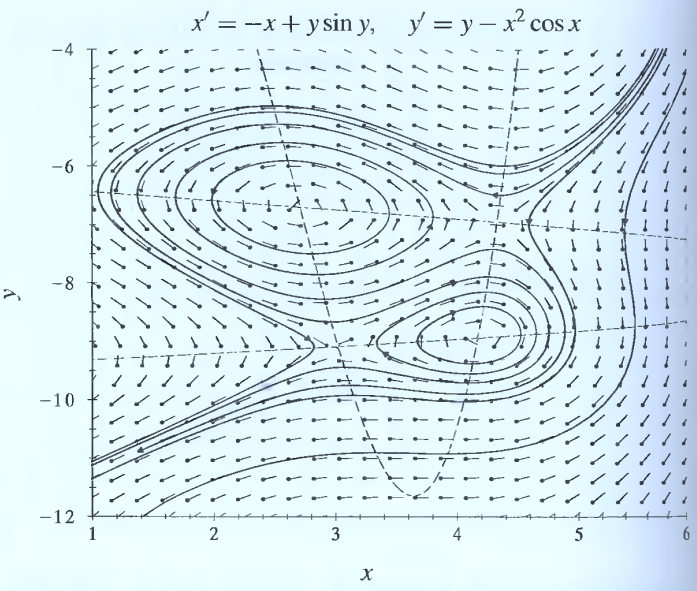
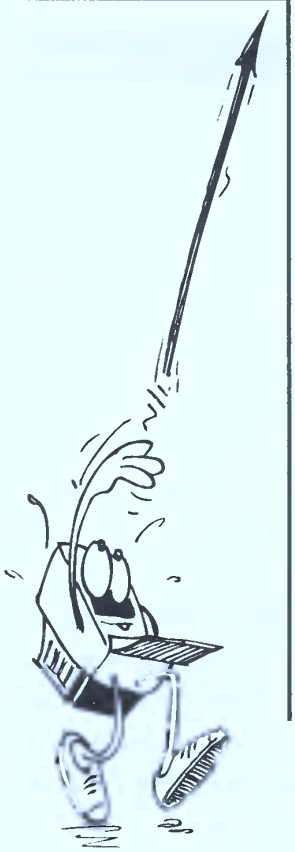


FIGURE 7.2.8 Some orbits are cycles, other orbits enter the computer screen and then leave as t increases (Example 7.2.5).

System (6) is linear, so we can solve it explicitly. The next system is neither linear, nor solvable in terms of elementary functions, but the direction field and nullcline approach still give us information about orbital behavior.

EXAMPLE 7.2.5



An Intriguing Direction Field: Nullclines, Orbits

The direction field and the nullclines in Figure 7.2.7 for the system

$$x' = -x + y\sin y$$

$$y' = y - x^2\cos x$$

shows something unusual happening near four equilibrium points. The plotted orbits of Figure 7.2.8 verify our suspicions. Two of the equilibrium points turn approaching orbits away. Nested cycles surround the other two equilibrium points.

See how orbits change direction as they cross the nullclines $x = y\sin y$ and $y = x^2\cos x$. The nullclines divide the rectangle, $1 \le x \le 6$, $-12 \le y \le -4$, into regions, inside each of which x' and y' have fixed signs. As an orbit moves horizontally across a y-nullcline (or vertically across an x-nullcline), y' (or x') changes sign. For example, both x' and y' are negative in the region at the upper right, so orbits fall to the left as t increases. In particular, the orbit in Figure 7.2.8 that enters the screen at $x = 6$, $y = -4.5$, moves down and to the left, crosses an x-nullcline vertically, enters a region where x' is positive, continues to fall (because $y' < 0$), and moves to the right until it crosses another arc of the x-nullcline, and so on. Eventually, this orbit exits the screen through the point $(1.42, -12)$.

We can analyze planar autonomous systems using direction fields and nullclines, but these techniques are not available for nonautonomous planar systems or for any system with more than two state variables.