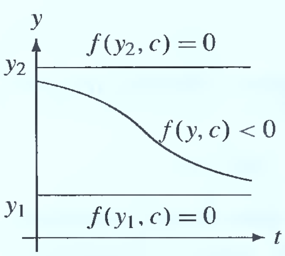
2.9 Bifurcations: A Harvested Logistic Model

For certain types of rate functions, we can identify the events that will cause radical qualitative changes in the long-term behavior of solutions of the corresponding ODE. These are *bifurcation events*. To recognize where the changes occur, we use an analytic approach based, not on a solution formula, but on the rate function itself.

Bifurcation: Sensitivity at the Edges

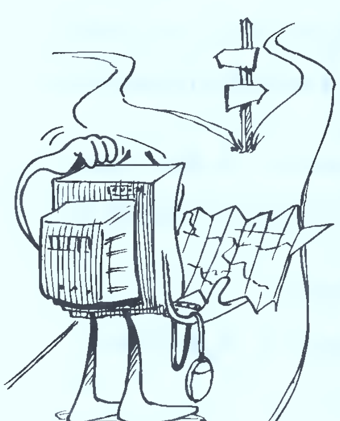
Equilibrium solutions play a critical role in determining the behavior of all solutions of a first-order *autonomous* ODE. Let's see what happens to the equilibria when the rate function depends on a parameter c:



$$y' = f(y, c) \tag{1}$$

For each value of c, the equilibrium solutions of ODE (1) are the zeros of the rate function $f(y, c)$. The equilibrium solution curves are straight lines that divide the ty-plane into horizontal bands. Inside each band, f has a fixed sign, and all solution curves rise or fall with time's advance away from one of the bounding lines and toward the other (see the margin figure). If c changes a little, then the bands will widen or narrow a little, but one expects the general appearance of the solution curves to be much as before. This expectation is reasonable, but often wrong.

As we change the value of c, the equilibrium solutions of ODE (1) change. At critical c-values, an equilibrium solution may split into two or more equilibrium solutions (i.e., *bifurcate*), merge with another, or even disappear entirely. After a bifurcation occurs, the long-term behavior of nonequilibrium solutions may be drastically altered. In a natural process modeled by ODE (1), the parameter c may be externally controlled by the environment, so natural conditions may trigger a bifurcation and cause the natural process to undergo a change in character over the long term. Tracking these changes as the parameter c changes is *bifurcation analysis*.



Procedure: Bifurcation Analysis of the ODE $y' = f(y, c)$

The three steps of a *bifurcation analysis* of the ODE $y' = f(y, c)$ are:

Part 1: Track the equilibrium solutions as they move, merge, split up, or disappear with changes in c.

Part 2: Describe the qualitative effects of these changes in c on the long-term behavior of nonequilibrium solutions.

Part 3: Summarize solution behavior as c changes in a bifurcation diagram.

Many patterns of bifurcation can occur when the parameter c in ODE (1) moves along the c-axis. We present two types in this section: *saddle-node bifurcation* and *pitchfork bifurcation*. At a saddle-node bifurcation an equilibrium point bifurcates to a repeller (the saddle) and an attractor (the node). The choice of the term "pitchfork" becomes clear at the end of this section. Let's start with the saddle-node type.

Saddle-Node Bifurcation: Harvesting/Restocking a Population

Ocean fishing is now under intense scrutiny because it is believed that over-fishing has brought stocks of several species of food fish such as cod (in the Atlantic) and salmon (in the Pacific) to dangerously low levels. How can we remedy this situation? Three strategies are currently being tested: lower the allowable limit of fish caught (fishermen don't like this at all), restrict fishing to a fixed season each year (acceptable to most fishermen, but with considerable grumbling), or develop ways to restock the fish population (fishermen prefer this approach). The decline of the fish populations is not just because of over-fishing; the pollution of streams, rivers, and the ocean itself is also a major factor. The bifurcation model outlined below is one starting point for thinking about the long-term effects of various harvesting and restocking policies.

A simple model for a logistically changing population undergoing constant-rate harvesting or restocking is

$$P' = r\left(1 - \frac{P}{K}\right)P + Q, \qquad P(0) = P_0 \tag{2}$$

where $P(t)$ is the population at time t and $r > 0$, $K > 0$, $P_0 \geq 0$, and Q are constants. The population is being harvested if Q is negative, restocked if Q is positive. What happens to the population levels if the *harvesting/restocking rate* Q is changed? Are there critical values of Q at which the long-term population levels undergo a dramatic change? We answer these questions for the case $r = 1$, $K = 1$. Later in the section, we show that we can always transform the general model IVP (2) to this case. Renaming P as y and Q as c, we consider the IVP

$$y' = (1 - y)y + c, \qquad y(0) = y_0 \tag{3}$$

Let's do a bifurcation analysis for the problem modeled by IVP (3). First, let's see how the equilibrium populations depend on the harvesting parameter c.

Bifurcation Analysis: Part 1

☞ The quadratic formula says that the roots of the polynomial $r^2 + ar + b$ are $r_1 = (-a + \sqrt{a^2 - 4b}\,)/2$ and $r_2 = (-a - \sqrt{a^2 - 4b}\,)/2$.

The zeros of the rate function in IVP (3) are the equilibrium populations. Use the quadratic formula to show that the rate function $f = (1 - y)y + c = -[y^2 - y - c]$ has zeros y_1, y_2 given by

$$y_1 = 0.5 - 0.5(1 + 4c)^{1/2}, \qquad y_2 = 0.5 + 0.5(1 + 4c)^{1/2} \tag{4}$$

See Figure 2.9.1 for a plot of $f(y, c)$ for various values of c. Using formulas (4),

- There are no real equilibria if $c < -0.25$. (The equilibria are complex numbers.)
- A single equilibrium ($y = 0.5$) appears for $c = -0.25$.
- There are two equilibria (both real) if $c > -0.25$.

So the bifurcation event occurs as c increases through the value -0.25. This completes the first part of our bifurcation analysis, which is an analysis of the zeros of f.

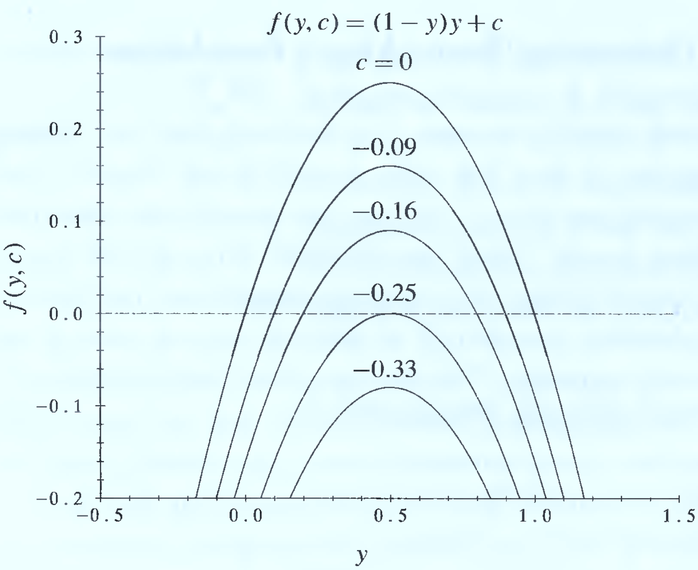


FIGURE 2.9.1 Saddle-node bifurcation: plots of $f(y, c)$, five values of c: bifurcation at $c = -0.25$.

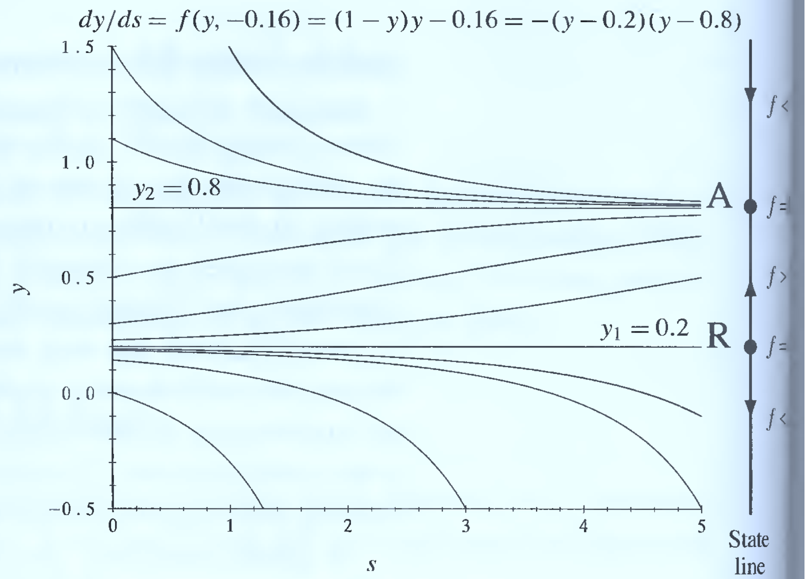


FIGURE 2.9.2 Saddle-node bifurcation: after the bifurcation; $c = -0.16$, light harvesting with equilibria at $y_1 = 0.2$ and $y_2 = 0.8$.

The sudden appearance of an equilibrium point and its splitting into two as the parameter c crosses a critical value is an example of a *saddle-node bifurcation*. It is one of a class of *tangent bifurcations*. They are called that because at the value of c where the bifurcation occurs, the graph of $f(y)$ in the yf-plane is tangent to the y-axis (see, e.g., the inverted parabola corresponding to $c = -0.25$ in Figure 2.9.1).

Now let's see how the population curves behave as we slide the parameter c through -0.25 upward to the value 0.

Bifurcation Analysis: Part 2

Figure 2.9.1 gives a plot of the rate function $f(y) = (1 - y)y + c$ against y for various values of c. From this plot, we can read off the properties of the solution curves of $y' = (1 - y)y + c$ as c changes. Extinction always occurs if the harvesting rate c is less than -0.25. The rate function in this case is always negative. At $c = -0.25$, there is a single value $y = 0.5$ for which $f(y) = 0$, and hence ODE (3) has a single equilibrium line $y = 0.5$. Solution curves above this equilibrium line fall toward it and those below fall away and hit the extinction line $y = 0$. The rate function is still negative for all $y \neq 0.5$. So $c = -0.25$ is a risky scenario for the population because a disturbance could force the population below the equilibrium level and then extinction is inevitable.

Let's set the harvesting rate c at -0.16 (a value above the bifurcation level of $c = -0.25$). ODE (3) now has two equilibrium lines $y_1 = 0.2$ and $y_2 = 0.8$, since the rate function $(1 - y)y - 0.16$ factors to $-(y - 0.2)(y - 0.8)$. Population curves above $y_1 = 0.2$ approach the saturation equilibrium level $y_2 = 0.8$. Curves below $y_1 = 0.2$ fall away and eventually the population becomes extinct; that is, $y(t^*) = 0$ at some time t^* that depends on the value of y_0, $0 \leq y_0 \leq 0.2$. The solution curves in Figure 2.9.2 illustrate the varieties of solution behavior in this case. The lowest four of the solution

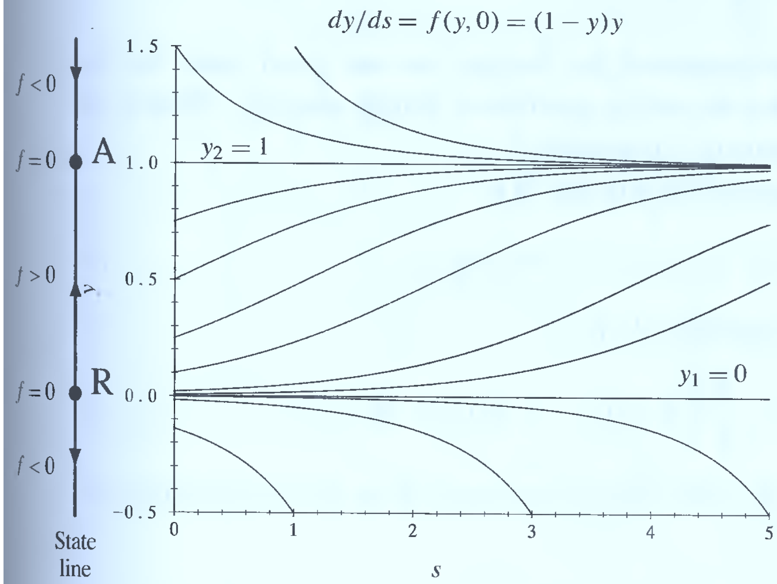


FIGURE 2.9.3 Saddle-node bifurcation: after the bifurcation; $c = 0$, all solution curves inside the population quadrant approach the saturation level $y = 1$.

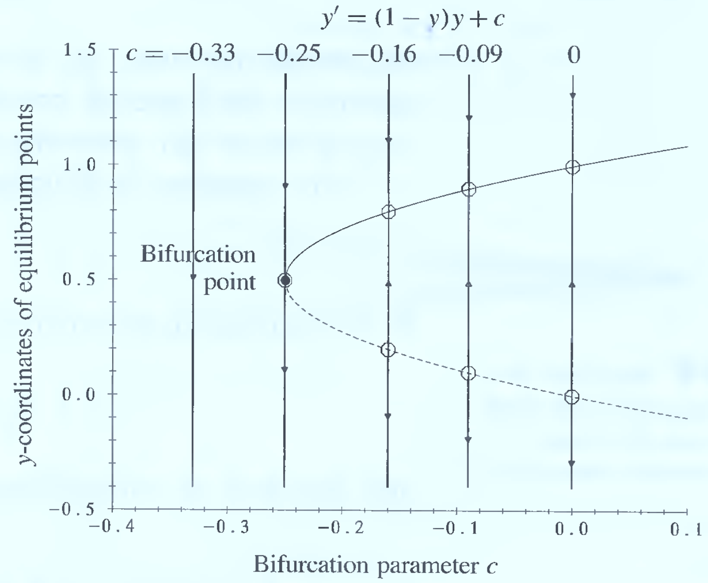


FIGURE 2.9.4 Saddle-node bifurcation diagram: state lines and the diagram for $y' = (1-y)y + c$.

☞ This qualitative approach is based on sign analysis (Section 2.8).

curves in the figure are curves of extinction. Solution behavior is encoded in the state line to the right of Figure 2.9.2. The point $y_1 = 0.2$ is a repeller on the state line and $y_2 = 0.8$ is an attractor.

If $c = 0$, there isn't any harvesting or restocking, and the equilibrium lines of ODE (3) are $y_1 = 0$ and $y_2 = 1$. Figure 2.9.3 shows these equilibrium lines and other solution curves. The upper line attracts nearby solution curves, but the lower line repels. The solution curves in Figure 2.9.3 below the line $y = 0$ have no physical meaning. The state line to the left of the figure encapsulates the solution behavior for $c = 0$; $y_1 = 0$ is a repeller and $y_2 = 1$ is an attractor. In this model, with no harvesting or restocking, any positive population will, over time, approach the carrying capacity of the ecosystem.

Now we will look at a different kind of diagram: a bifurcation diagram.

Bifurcation Analysis: Part 3

Let's summarize what we've learned so far in a *bifurcation diagram* (Figure 2.9.4). We graph the curves representing the location of the equilibrium points as c varies:

$$\text{(dashed) } y_1(c) = 0.5 - (0.5)(1+4c)^{1/2}, \quad \text{(solid) } y_2(c) = 0.5 + (0.5)(1+4c)^{1/2}$$

In this diagram we follow the convention that solid arcs denote attractors and dashed arcs denote repellers. Each point $(c, y_2(c))$ on the solid arc attracts all points on the state line that are above the repelling point $((c, y_1(c))$. The greater the vertical separation between the solid and the dashed arcs, the larger the region of attraction of the point $(c, y_2(c))$. The five vertical state lines shown in Figure 2.9.4 correspond to five different values of c.

This bifurcation diagram tells the story of bifurcation, harvesting, and logistic change. At the left of the diagram we see disaster for the population and, in the long run, for the harvester, but in the middle of the diagram we see good times for the population and harvester, provided the initial population is high enough. What's the story at the far right where the value of c is positive?

This completes the bifurcation analysis of the IVP

$$y' = (1-y)y + c, \qquad y(0) = y_0 \tag{5}$$

Now let's return to the original population IVP

☞ Recall that the population is harvested when $Q < 0$ and restocked when $Q > 0$.

$$P' = r\left(1 - \frac{P}{K}\right)P + Q, \qquad P(0) = P_0 \tag{6}$$

and show how we can transform it to the IVP (5) we used for the bifurcation analysis.

Scaling Population and Time

Let's transform IVP (6) to IVP (5) by *scaling* the time and population variables.

Suppose that

$$P = ay \qquad \text{and} \qquad t = bs$$

where a and b are positive *scaling constants* that we will choose strategically, and y and s are the new scaled population and time variables. Note that $dP/dy = a$ and $ds/dt = 1/b$. Inserting these changes into IVP (6) and using the Chain Rule, we have

$$\frac{dP}{dt} = \frac{dP}{dy}\frac{dy}{ds}\frac{ds}{dt} = \frac{a}{b}\frac{dy}{ds} = r\left(1 - \frac{a}{K}y\right)ay + Q, \qquad P(0) = P_0 = ay(0) \tag{7}$$

Multiply the ODE in (7) by b/a and the initial condition by $1/a$ to get

$$\frac{dy}{ds} = br\left(1 - \frac{a}{K}y\right)y + \frac{b}{a}Q, \qquad y(0) = \frac{P_0}{a} \tag{8}$$

We can simplify IVP (8) for the scaled population y by setting

$$a = K, \qquad b = \frac{1}{r}, \qquad c = \frac{b}{a}Q = \frac{Q}{rK}, \qquad y_0 = \frac{P_0}{a} = \frac{P_0}{K} \tag{9}$$

to obtain the transformed IVP

$$\frac{dy}{ds} = (1-y)y + c, \qquad y(0) = y_0 \tag{10}$$

with just the two parameters c and y_0, and not the four parameters r, K, Q, and P_0 of IVP (2). The parameter c plays the role of the harvesting/restocking term in the rescaled system.

Rescaling the variables has another very interesting feature: the scaled population and time variables are dimensionless. Here's why: suppose, for example, that the original variables P and t have the dimensions tons and years, respectively. Then

☞ $\dim K = \text{tons}$, $\dim r = \text{yr}^{-1}$.

$$\dim y = \dim \frac{P}{a} = \dim \frac{P}{K} = \frac{\text{tons}}{\text{tons}}$$

$$\dim s = \dim \frac{t}{b} = \dim tr = (\dim t)(\dim r) = \frac{\text{years}}{\text{years}}$$

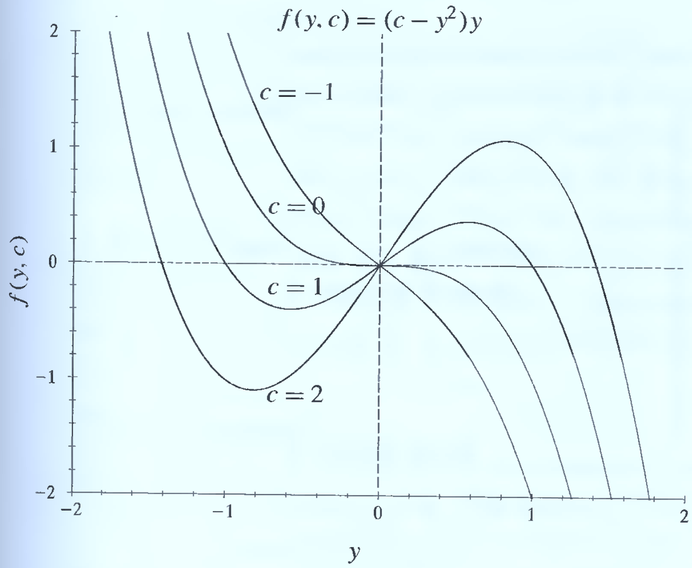


FIGURE 2.9.5 Pitchfork bifurcation: the graphs of $f(y, c)$ for four c-values with bifurcation at $c = 0$.

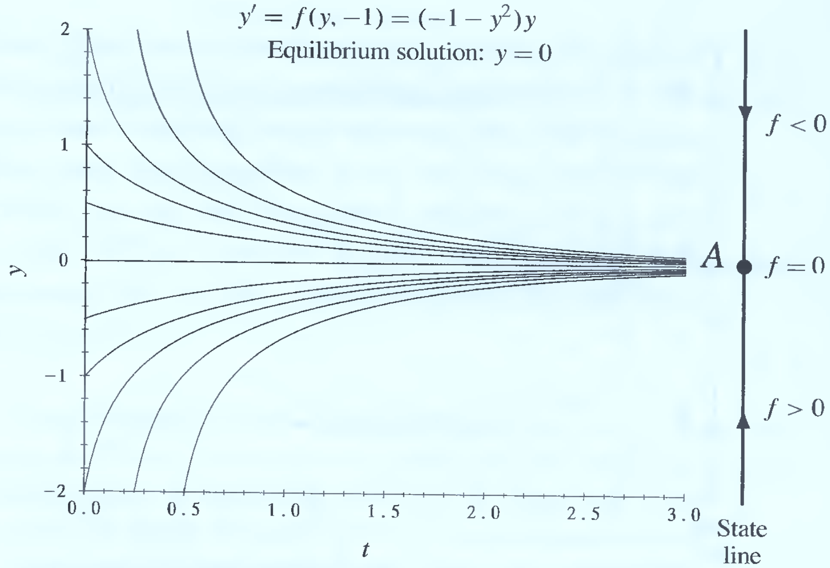
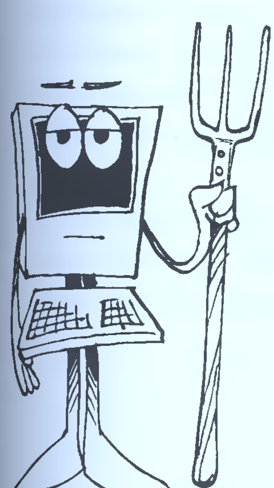


FIGURE 2.9.6 Pitchfork bifurcation: $c = -1$, before the bifurcation there is one real equilibrium.

so y and s are dimensionless. Similarly, the parameters c and y_0 are dimensionless. That means that any qualitative conclusions we reach about solutions of IVP (10) remain valid no matter what units are selected to measure population levels and time. The process of scaling variables to remove dimensions is *nondimensionalization.*

Finally, we see from the third equality in (9) that the bifurcation point $c = -0.25$ corresponds to the value $Q = -rK/4$, which defines the *critical harvesting rate* to be $-rK/4$ tons/yr. The analysis that we have done has profound implications for the harvested species. It implies that if the harvesting rate is *supercritical* (i.e., if Q, which is negative in this situation, is less than $-rK/4$), then the species is doomed. But if the harvesting rate is *subcritical* (so $Q > -rK/4$), then the species has a chance of survival and the *sustainable yield* for the harvester is $rK/4$ tons/yr.

Tangent Bifurcations: The Pitchfork



There are other kinds of tangent bifurcations that occur in natural processes, and the *pitchfork bifurcation* is one of them.

Here's an example of a pitchfork bifurcation. The nonlinear autonomous ODE

$$y' = (c - y^2)y \tag{11}$$

contains a parameter c. Let's carry out the three-part bifurcation analysis of ODE (11) as c varies.

Part 1. The equilibrium solutions of ODE (11) are given by

$$y_1 = 0, \quad y_2 = c^{1/2}, \quad \text{and} \quad y_3 = -c^{1/2} \tag{12}$$

For negative values of c the only visible equilibrium is y_1, while y_2 and y_3 are complex-valued. For $c = 0$, ODE (12) has only one equilibrium solution: $y = 0$. As the value of c increases through 0, the equilibrium $y = 0$ bifurcates into three real-valued equilibria: y_1, y_2, y_3, where $y_3 < y_1 = 0 < y_2$. Figure 2.9.5 shows graphs of the rate function

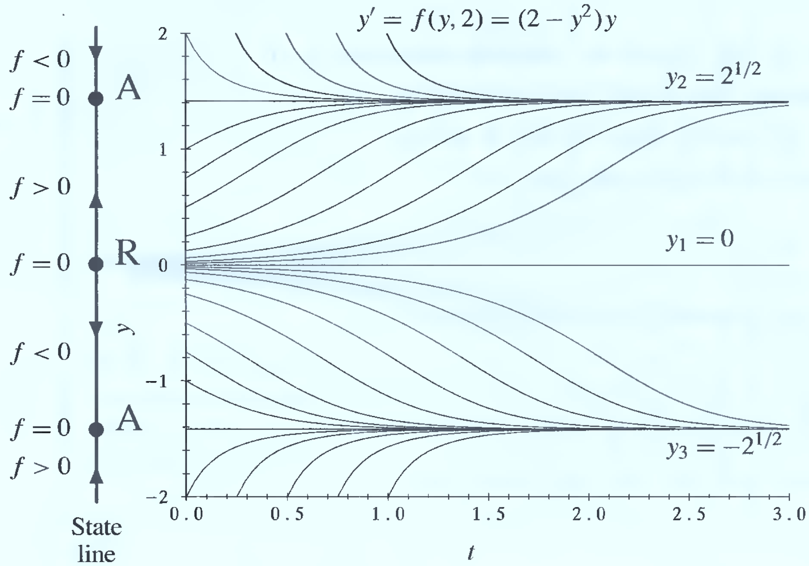


FIGURE 2.9.7 Pitchfork bifurcation: $c = 2$, after the bifurcation there are three real equilibria.

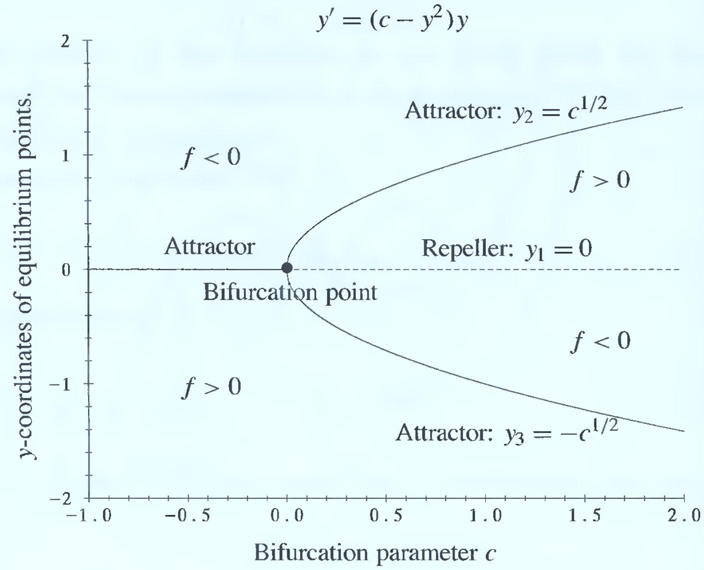


FIGURE 2.9.8 Pitchfork bifurcation diagram for $y' = (c - y^2)y$.

$f(y, c) = (c - y^2)y$ for $c = -1, 0, 1, 2$. For each positive value of c the graph of f cuts the y-axis at the three points $y_1 = 0$, $y_2 = c^{1/2}$, and $y_3 = -c^{1/2}$. At the bifurcation value $c = 0$, the graph of f is tangent to the y-axis at $y_1 = 0$, so this is another tangent bifurcation.

Part 2. Figure 2.9.6 shows equilibrium lines and other solution curves for $c = -1$ (before the bifurcation), and Figure 2.9.7 shows equilibrium lines and other solution curves for $c = 2$ (beyond the bifurcation). The equilibrium $y = 0$ is an attractor before the bifurcation, but it is a repeller after the bifurcation, having transferred its attracting character to the two new outlying equilibria $y = y_2$, y_3. The state lines alongside Figures 2.9.6 and 2.9.7 display the attracting and repelling behavior of the equilibrium points.

Part 3. Figure 2.9.8 is the *pitchfork bifurcation diagram* for ODE (11). It shows the equilibrium values as functions of the bifurcation parameter c. The solid arcs correspond to attracting equilibrium solutions, and the dashed line corresponds to repelling equilibrium solutions. For example, a point P on the lower parabolic arc attracts all points on a vertical line through P but below the dashed line $y_1 = 0$. We can see in Figure 2.9.8 the reason for the name "pitchfork bifurcation." We leave it to the reader to draw some vertical state lines in Figure 2.9.8.

☞ Refer back to Figure 2.9.4 for guidance.

It is no coincidence that at the c-value where bifurcation occurs, the graph of $f(y, c)$ in the yf-plane is tangent to the y-axis. This is true for both the saddle-node and the pitchfork bifurcations. That kind of tangent behavior is usually the clue that a bifurcation of some kind has occurred.

Looking Back

If we want to see what happens to solutions of an ODE over a long span of time, then we have to take into account the fact that small changes in the data may in time cause large changes in the solutions. The radical changes in the behavior of solution curves as a parameter goes through a bifurcation value are clearly observable, but only after a long span of time. This is because solutions change continuously with the data, and small changes in the data mean only small changes in the solutions over a short time span. There is no contradiction between the concept of continuity in the data (see the SPOTLIGHT ON CONTINUITY IN THE DATA) and the marked changes in solution behavior seen after a bifurcation event. The former is short-term behavior, the latter has to do with behavior far into the future.

PROBLEMS

Saddle-Node Bifurcations. Explain why there is a saddle-node bifurcation at some value of the parameter c for each ODE below. In each case sketch the saddle-node bifurcation diagram, using solid arcs for attracting equilibria and dashed arcs for repelling equilibria. [*Hint*: see the discussion about Figure 2.9.4.]

1. $y' = c - y^2$ **2.** $y' = c - 2y + y^2$ **3.** $y' = c + 2y + y^2$

Solution Curves of Saddle-Node Bifurcations. Plot state lines and several solution curves of the ODEs in Problems 1, 2, 3 for each of several values of c above, at, and below the saddle-node bifurcation value. Describe the behavior of the curves in each case as t increases.

4. Problem 1 **5.** Problem 2 **6.** Problem 3

Pitchfork Bifurcations. Explain why there is a pitchfork bifurcation at some value of the parameter c for each ODE. In each case sketch the pitchfork bifurcation diagram, using solid arcs for attracting equilibria, dashed arcs for repelling equilibria. [*Hint*: see the discussion about Figure 2.9.8.]

7. $y' = (c - 2y^2)y$ **8.** $y' = -(c + y^2)y$ **9.** $y' = (c - y^4)y$

Solution Curves of Pitchfork Bifurcations. Plot state lines and several solution curves of the ODEs in Problems 7, 8, 9 for values of c above, at, and below the pitchfork bifurcation value. Describe the behavior of the curves in each case as t increases.

10. Problem 7 **11.** Problem 8 **12.** Problem 9

Yet another kind of bifurcation.

Transcritical Bifurcations. In a *transcritical bifurcation*, as the parameter c in the rate function for the ODE $y' = f(y, c)$ is changed, a pair of equilibrium solutions, one an attractor and the other a repeller, merge and then separate, exchanging their attracting or repelling properties in the process. Explain why each of the following ODEs has a transcritical bifurcation. Draw a bifurcation diagram (solid arcs for attracting equilibria, dashed arcs for repelling equilibria). Then plot state lines and several solution curves for the ODEs for values of c above, at, and below the bifurcation values.

13. $y' = cy - y^2$ **14.** $y' = cy + 10y^2$

Modeling Problems.

15. *Too Late to Save a Population from Extinction?* Referring to the text and the discussion of the harvested population model ODE, $P' = r(1 - P/K)P - H$, $H > rK/4$, explain why the following is true: if the value of P is near 0, then restricting the harvest rate to slightly below the critical harvesting rate $rK/4$ will not save the population from extinction. What if you ban harvesting altogether in this case? Can you save the species?

16. *How Many Hunting Licenses Should Be Issued?* The duck population near a hunting lodge is modeled by the ODE, $P' = (1 - P/1000)P - H$, where H is the harvesting rate.

(a) How many licenses can be issued per year so that the duck population has a chance of survival? Each hunter is allowed to shoot up to 20 ducks per year.

(b) Suppose N licenses are issued, where N is less than the maximal number found in **(a)**. What values of the initial duck population lead to total extinction of the species? Explain.