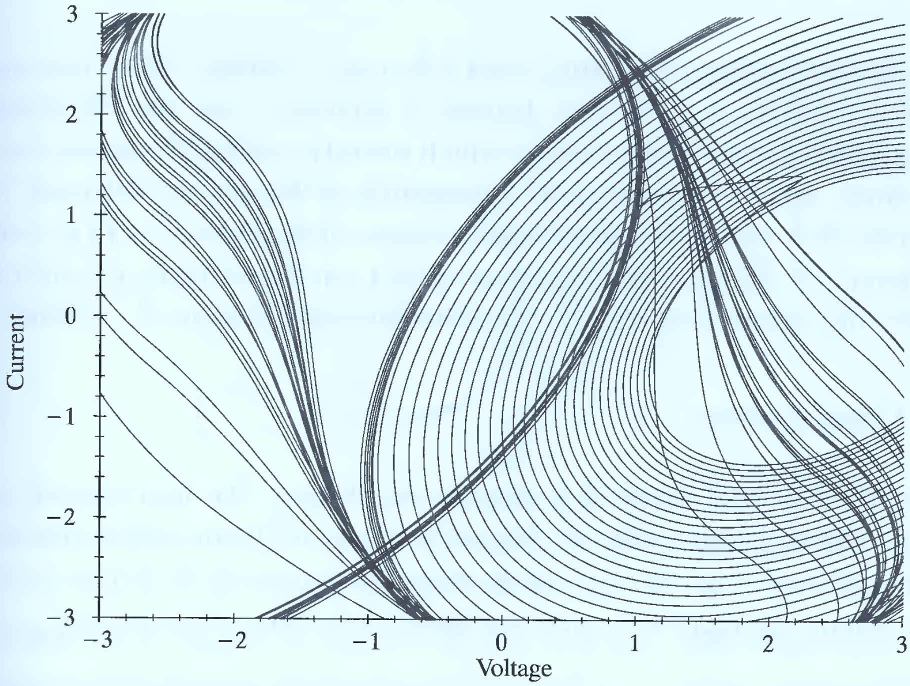


Chaotic current and voltage in a nonlinear circuit? Take a look at the scroll circuit in the WEB SPOTLIGHT ON CHAOS IN A NONLINEAR CIRCUIT.

CHAPTER 9

Nonlinear Systems: Cycles and Chaos

In this chapter we look at the long-term behavior of orbits of systems of ODEs (usually autonomous and nonlinear, often planar). We will see orbits attracted to a cycle (an orbit of a periodic solution) or to a cycle-graph (a strange combination of equilibrium points and orbits on a closed graph). We will see bounded orbits that wander chaotically between regions of oscillatory behavior. We will also see unusual behavior that suddenly appears as a parameter is pushed beyond a bifurcation point. The material in this chapter takes us to the edge of understanding of dynamical behavior.

9.1 Cycles

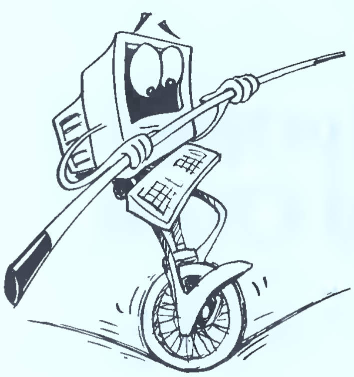
...the aeolian harp, a pneumatic hammer, the scratching noise of a knife on a plate, the waving of a flag in the wind, the humming noise sometimes made by a water-tap, ..., the periodic recurrence of epidemics and of economic crises, the periodic density of an even number of species of animals living together and the one species serving as food for the other ..., and finally, the beating of a heart.

–Balthazar van der Pol

Each of these phenomena has a repeating event at its core. Ordinary differential equations model these systems with varying degrees of accuracy. Van der Pol designed circuits that sustained currents and voltages which steadily oscillated without any periodic driving force, and he recognized the universality of this kind of behavior.[1] According to van der Pol, nonlinear autonomous systems of the type $x' = F(x)$ (where $F(x)$ is a continuously differentiable function vector) can model many physical systems that exhibit this kind of behavior. So, the Fundamental Theorem 7.1.1 holds.

Cycles and Limit Cycles

A *cycle* is the orbit in state space of a periodic oscillation. We have already seen many cycles, for example, the cycles of a linear harmonic oscillator such as the model of an undamped Hooke's Law Spring and the nonlinear cycles of the Lotka–Volterra predator-prey system. We focus now on a special kind of cycle, a limit cycle.



❖ **Limit Cycle.** A cycle of an autonomous system is a *limit cycle* if some nonperiodic orbit tends to it as $t \to +\infty$ or as $t \to -\infty$. A limit cycle is an *attractor* if every nearby orbit approaches it as $t \to +\infty$, and a *repeller* if every nearby orbit approaches it in reverse time, that is, as $t \to -\infty$.

The cycles of the linear harmonic oscillator (Section 3.4) and of the nonlinear Lotka–Volterra system (Section 2.6) are *not* limit cycles because all orbits near each of these cycles are also cycles. This prevents them from approaching one another as $t \to +\infty$ or as $t \to -\infty$. In this section and the next we will search for limit cycles of planar autonomous systems $x' = f(x, y)$, $y' = g(x, y)$, where f and g are continuously differentiable.

EXAMPLE 9.1.1

Polar Coordinates and Cycles

Let's show that the following system has an attracting limit cycle:

$$x' = x - y - x(x^2 + y^2), \qquad y' = x + y - y(x^2 + y^2) \tag{1}$$

We use polar coordinates to conveniently describe the orbits. Differentiate $r^2 = x^2 + y^2$ and $\tan\theta = y/x$ via the chain rule:

☞ Refer to Section 7.2, formula (12), for another way to transform a planar system from rectangular to polar coordinates.

$$\begin{aligned} 2rr' &= 2xx' + 2yy' = 2x[x - y - x(x^2 + y^2)] + 2y[x + y - y(x^2 + y^2)] \\ &= 2(x^2 + y^2) - 2(x^2 + y^2)^2 = 2r^2 - 2r^4 \\ (\sec^2\theta)\theta' &= \frac{xy' - yx'}{x^2} = \frac{1}{x}\left[x + y - y(x^2 + y^2)\right] - \frac{y}{x^2}\left[x - y - x(x^2 + y^2)\right] \\ &= 1 + \left(\frac{y}{x}\right)^2 = \frac{x^2 + y^2}{x^2} = \sec^2\theta \end{aligned}$$

[1]The Dutch physicist and engineer Balthazar van der Pol (1889–1959) did pioneering work on the first commercially available radios in the 1920s. His mathematical models for the behavior of a radio's internal currents and voltages are still in use, and we will look at one of them later in this section.

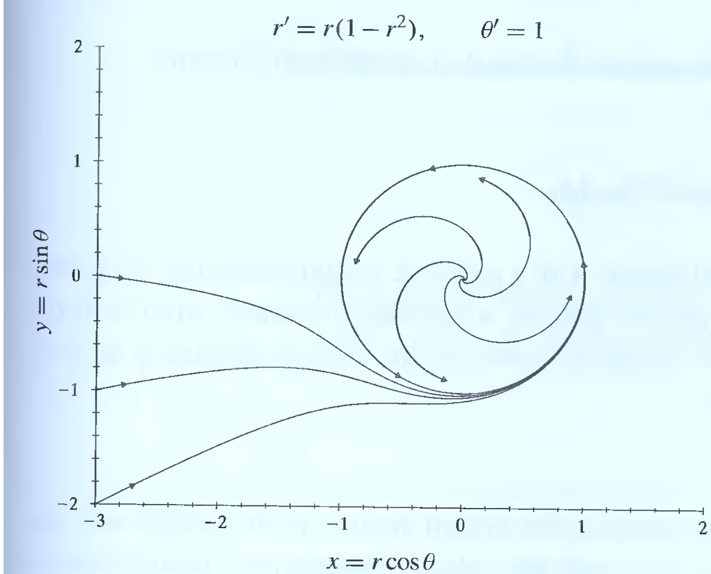


FIGURE 9.1.1 Attracting limit cycle at $r = 1$ encloses an unstable equilibrium point (Example 9.1.1).

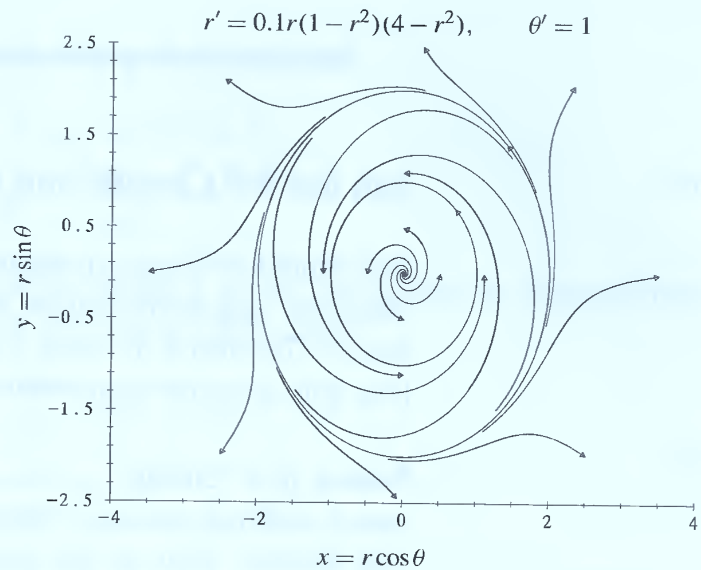


FIGURE 9.1.2 Attracting and repelling limit cycles enclose an unstable origin (Example 9.1.2).

Cancel common factors and obtain the system in polar coordinates equivalent to (1):

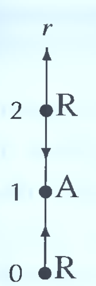
$$r' = r(1 - r^2), \qquad \theta' = 1 \tag{2}$$

Antidifferentiate the θ' equation to obtain $\theta(t) = t + C$. So $\theta(t + 2\pi) = \theta(t) + 2\pi$, and orbits move counterclockwise around the equilibrium point of system (1) at the origin in a full rotation every 2π units of time. The ODE for r in (2) tells us that $r = 0$ and $r = 1$ are solutions. The solution $r = 0$ corresponds to the equilibrium point at the origin, while $r = 1$ corresponds to the circular cycle $x^2 + y^2 = 1$ for system (1). If $0 < r(0) < 1$, then sign analysis of the ODE $r' = r(1 - r^2)$ tells us that $r(t)$ decreases to zero as $t \to -\infty$ and increases to 1 as $t \to +\infty$. For $r_0 > 1$, $r(t)$ decreases to 1 as $t \to +\infty$ and increases to $+\infty$ as t decreases (see the state line in the margin). The unit circle is thus a limit cycle of period 2π that attracts (A) all nonconstant orbits of system (1), but the origin repels (R). See Figure 9.1.1 for eight orbits.

In the next example, we go directly to a system in polar coordinates and use sign analysis to examine the qualitative behavior of its orbits.

EXAMPLE 9.1.2

Two Limit Cycles: One Attracts, the Other Repels



The system in polar coordinates

$$r' = 0.1r(1 - r^2)(4 - r^2), \qquad \theta' = 1$$

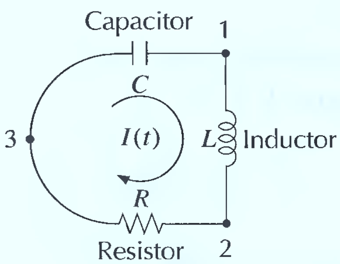
has a repelling equilibrium point ($r = 0$), an attracting limit cycle ($r = 1$), and a repelling limit cycle ($r = 2$). The sign changes of r' as r passes through $r = 1$ and $r = 2$ suggest these results (see the state line in the margin). As in the previous example, each cycle has period 2π, and orbits turn counterclockwise because θ' is positive (Figure 9.1.2). We inserted the factor of 0.1 in the rate function for r to make the graphs in Figure 9.1.2 attractive (try some other factors and see what happens).

Let's look at the planar system van der Pol used to model radio circuits.

Van der Pol Circuits and Limit Cycles

How would we design an electrical circuit with a periodic output current of prescribed amplitude and period toward which the current would return quickly after a disturbance? One idea is to create a new circuit by replacing the passive resistor in an *RLC* loop with an active semiconductor.

Passive *RLC* Circuit



Attach and then remove a voltage source to the circuit shown in the margin and then take it away. How do the current $I(t)$ and the voltages across the circuit elements change with the passage of time? According to Kirchhoff's Voltage Law (Section 4.4), the voltages across the three circuit elements satisfy

$$V_{13} = V_{12} + V_{23} \tag{3}$$

where V_{ij} denotes the voltage drop from node i to node j. Faraday's Law, Ohm's Law and the differential form of Coulomb's Law, respectively, relate the voltages to the corresponding circuit elements, where L, R, and C are positive constants:

☞ Look back at Section 4.4 for these laws.

$$\begin{aligned} V_{12} &= LI' \\ V_{23} &= RI \\ V_{13}' &= -\frac{1}{C}I \end{aligned} \tag{4}$$

The minus sign in the last equation follows from the relation $V_{13} = -V_{31}$. The two state variables I and $V = V_{13}$ characterize the dynamics of the circuit via the linear system

☞ From solutions $I(t)$ and $V(t) = V_{13}(t)$ of (5), use (3) and (4) to obtain V_{23} and V_{12}.

$$\begin{aligned} I' &= \frac{1}{L}V_{12} = \frac{1}{L}(-V_{23} + V) = \frac{1}{L}(-RI + V) \\ V' &= -\frac{1}{C}I \end{aligned} \tag{5}$$

The system matrix for (5) has the characteristic polynomial $\lambda^2 + R\lambda/L + 1/LC$. Since the characteristic roots have negative real parts, the current $I(t)$ and the voltage $V(t)$ tend to zero as $t \to +\infty$. There are no periodic solutions, so there are no cycles. The physical reason for this decay is that the resistor dissipates electrical energy.

Active *RLC* Circuit

The circuit is active if energy is pumped in whenever the amplitude of the current is too low. Replace the resistor by an active element that acts as a negative resistor at low current levels but dissipates energy at high levels. Now modern semiconductor devices do just this.

☞ A negative resistor pumps energy into the circuit.

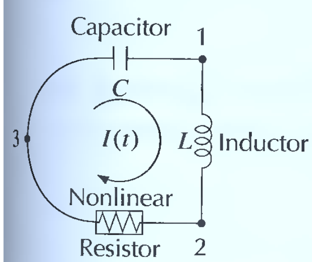
We obtain the rate equations for the active circuit from the passive circuit system (5) when we replace the Ohm's Law voltage drop RI by a nonlinear function

$F(I)$ that models the behavior of the active circuit element:

$$I' = \frac{1}{L}(-F(I) + V)$$
$$V' = -\frac{1}{C}I \quad (6)$$

If we rescale time t, current I, and the voltage V across the capacitor to dimensionless form, system (6) takes on the form of a *van der Pol system*

$$\frac{dx}{d\tau} = y - \mu f(x)$$
$$\frac{dy}{d\tau} = -x \quad (7)$$

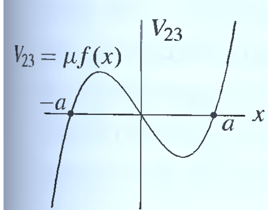


where

$$\tau = t/\sqrt{LC}, \qquad x = I\sqrt{L}, \qquad y = V\sqrt{C}$$

$$\mu = \sqrt{C}, \qquad f(x) = F(x/\sqrt{L})$$

We continue to refer to the new dimensionless variables τ, x, and y as, respectively, time, current, and voltage.



The margin figures show an active *van der Pol circuit* and the nonlinear voltage-current relationship of a typical semiconductor device. In particular, note that voltage y and current x have opposite signs if $|x| < a$. The nonlinear nature of the semiconductor is enough to completely change the long-term behavior of the current in the circuit and the voltage across the capacitor.

Here is the mathematical result based on van der Pol's work:

THEOREM 9.1.1

The van der Pol Cycle

Suppose that the continuous and piecewise smooth function $f(x)$ has the properties: **(a)** $f(-x) = -f(x)$; **(b)** for some positive constant a, $f(x)$ is negative for $0 < x < a$, but positive for $x > a$; and **(c)** $f(x) \to +\infty$ as $x \to +\infty$. Then for each positive value of μ the van der Pol system $x' = y - \mu f(x)$, $y' = -x$ has a unique limit cycle that encloses the equilibrium point $x = 0$, $y = 0$. This cycle attracts all nonconstant orbits.

☞ We sketch the proof is in the Student Resource Manual.

Let's see what the limit cycles look like.

EXAMPLE 9.1.3

A Particular Van der Pol System

Van der Pol originally used the function

$$f(x) = \frac{1}{3}x^3 - x \quad (8)$$

which meets the conditions of Theorem 9.1.1 with $a = \sqrt{3}$. Figures 9.1.3 and 9.1.4 show the x-nullcline, $y = \mu(x^3/3 - x)$ for $\mu = 1, 20$, and two orbits that wind toward the attracting limit cycle (one from the inside and one from the outside). The figures also show the corresponding x- and y-component graphs. For large enough values of

☞ The two orbits in Figure 9.1.4 approach the limit cycle so fast that what you see in the figure is a good approximation of the limit cycle.

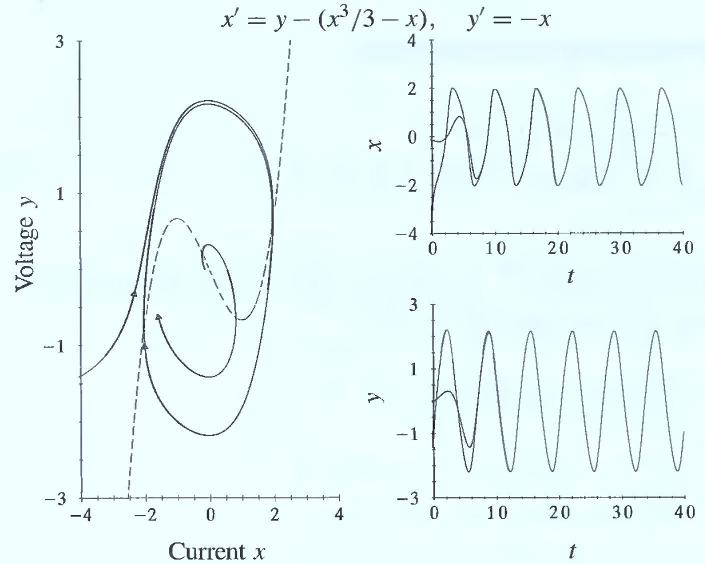


FIGURE 9.1.3 $\mu = 1$: two orbits approach the limit cycle. The x-nullcline is the dashed curve (Example 9.1.3).

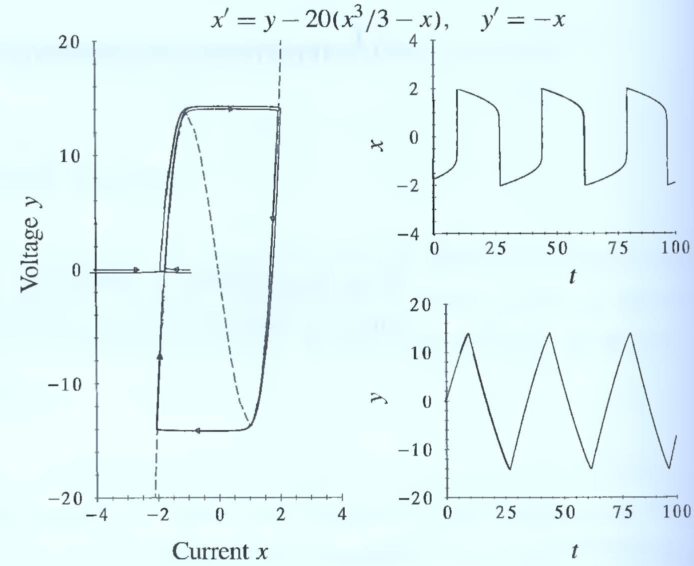


FIGURE 9.1.4 $\mu = 20$: two orbits approach the limit cycle The x-nullcline is the dashed curve. (Example 9.1.3).

μ, the limit cycle has slanting sides nearly along arcs of the x-nullcline. For these values of μ, the top and the bottom of the cycle correspond to rapid changes in the magnitude of the current x (see the nearly vertical segments of the x-component graph in Figure 9.1.4). Slow changes along the sides of the cycle result as the current relaxes and the voltage reverses its sign. This is why the limit cycle of a van der Pol system for large values of μ is a *relaxation oscillation*.

☞ The period 6.5 for $\mu = 1$ is close to 2π. That's not surprising since all nonconstant orbits are sinusoids with a period of 2π for $\mu = 0$.

We can tune the circuit by changing the value of μ. This alters the period of the cycle, but it has little effect on the amplitude of the current. For example, we see from the graphs in Figures 9.1.3 and 9.1.4 that the period is approximately 6.5 units of dimensionless time τ if $\mu = 1$ and about 33 units if $\mu = 20$. In each graph, the dimensionless current has amplitude 2. In contrast to the current, the amplitude of the dimensionless voltage y across the semiconductor does increase with μ.

The Jacobian matrix at the origin for the van der Pol system $x' = y - \mu(x^3/3 - x)$, $y' = -x$ is $J = \begin{bmatrix} \mu & 1 \\ -1 & 0 \end{bmatrix}$ with eigenvalues $[\mu \pm (\mu^2 - 4)^{1/2}]/2$. Since μ is positive, the eigenvalues are real and positive if $\mu \geq 2$, but they are complex conjugates with positive real parts if $0 < \mu < 2$. By the results of Section 6.5 we know that the origin is a repelling and unstable equilibrium point, so we conclude that nearby orbits move outward from the origin as time advances.

Looking Back

Van der Pol's limit cycles and the limit cycles of the examples are internally generated by the dynamics of the corresponding autonomous systems and are *not* the response to some external periodic driving force. It is no accident that these systems are nonlinear. Indeed, an autonomous linear system $x' = Ax$ has *no* attracting and *no* repelling limit cycles. For, if $x = x(t)$ is a periodic solution of $x' = Ax$, then by linearity so is $x =$

$cx(t)$ for all real constants c. This means that any cycle of a linear system is part of a family of cycles, so it can't attract (or repel) all nearby orbits.

Cycles of a planar autonomous system have two distinctive properties:

- Each cycle encloses one or more equilibrium points.[2]
- Each cycle divides the plane into two regions (i.e., the interior and exterior regions of the cycle), and no orbit that starts in one region can penetrate the other without violating uniqueness.

The first property tells us that if we want to find cycles, we should locate the equilibrium points first. The second property implies that we can regard the dynamics and orbital behavior in the interior region bounded by a cycle as independent from that in the exterior region, and vice versa.

PROBLEMS

Cycles and Limit Cycles. Find the equilibrium points and the cycles of the following systems written in polar coordinates. As in Examples 9.1.1 and 9.1.2, draw labeled state lines for r. Sketch the cycles and other orbits in the xy-plane, $x = r\cos\theta$, $y = r\sin\theta$. Use arrowheads to show the direction of increasing time. Is the equilibrium point at the origin asymptotically stable, neutrally stable, or unstable? Determine whether each cycle is a limit cycle, and if it is, whether it attracts or repels.

1. $r' = 4r(4-r)(5-r)$, $\theta' = 1$
2. $r' = r(r-1)(2-r^2)(3-r^2)$, $\theta' = -3$
3. $r' = r(1-r^2)(4-r^2)$, $\theta' = 1-r^2$ [*Hint*: r' and θ' are 0 if $r = 1$.]
4. $r' = r(1-r^2)(9-r^2)$, $\theta' = 4-r^2$ [*Hint*: $\theta' = 0$ but $r' \neq 0$ if $r = 2$.]
5. $r' = r\cos\pi r$, $\theta' = 1$

Limit Cycles. Find all limit cycles and identify each as an attractor or a repeller. Use polar coordinates as in Example 9.1.1 and draw labeled state lines for r.

6. $x' = y - x(x^2+y^2)$, $y' = -x - y(x^2+y^2)$
7. $x' = x + y - x(x^2+y^2)$, $y' = -x + y - y(x^2+y^2)$
8. $x' = 2x - y - x(3 - x^2 - y^2)$, $y' = x + 2y - y(3 - x^2 - y^2)$
9. $r' = r(1-r^2)(4-r^2)(9-r^2)/1000$, $\theta' = 1$. Plot orbits in the rectangle $|x| \le 6$, $|y| \le 4$, where $x = r\cos\theta$, $y = r\sin\theta$.

Center-Spiral Equilibrium Points.

10. *Center-Spiral* Consider the system in polar coordinates $r' = r^3\sin(1/r)$, $\theta' = 1$, where r' is defined to be zero at $r = 0$. Show that the corresponding nonlinear xy-system has infinitely many circular limit cycles around the neutrally stable equilibrium point at the origin, the sequence of shrinking cycles converges onto the origin, the cycles alternately attract and repel, and all other orbits spiral away from one cycle and toward another as t increases. The origin of the xy-system is a *center-spiral*. Explain the name. [*Hint*: plot xy-orbits in an $r\cos\theta$, $r\sin\theta$ plane.]

[2]For a proof, see page 252 of *Differential Equations, Dynamical Systems, and Linear Algebra* by Morris W. Hirsch and Stephen Smale (New York: Academic Press, 1974).

11. Explain why the origin of the xy-plane is a center-spiral equilibrium point for the system $r' = r\sin(\pi/r)$, $\theta' = -2$, where r' is defined to be zero if $r = 0$.

Unusual Cycles.

12. *Nonisolated, Nonlinear Cycles and a Center* Show that the nonconstant orbits of the system $x' = y^3$, $y' = -x^3$ are cycles that enclose the equilibrium point at the origin and fill the xy-plane. Plot the orbits and component curves corresponding to initial points $(0, 0)$, $(0.5, 0)$, $(1, 0)$, $(2, 0)$. Are the periods of distinct cycles the same?

13. *Semistable Cycles* Some cycles repel on one side and attract on the other; they are one kind of *semistable* cycle. Find all cycles of $r' = r(1 - r^2)^2(4 - r^2)(9 - r^2)$, $\theta' = 1$ and identify each as attracting, repelling, or semistable. Sketch orbits and draw a labeled state line for r.

14. *A Strange Cycle* Explain why the system $r' = r(r - 1)^2 \sin[\pi/(r - 1)]$, $\theta' = 1$, where r' is defined to be zero if $r = 1$, has a cycle $r = 1$, which is not a limit cycle, and every neighborhood of which contains other cycles as well as spirals between successive cycles. [*Hint*: as $r \to 1$, $(r - 1)^2 \sin[\pi/(r - 1)] \to 0$; $r' = 0$ at $r = 1$ and at $r = 1 \pm 1/n$.]

15. *Cycles in Space* Describe the orbits near the cycle $r = 1$, $z = 0$ of the system (in cylindrical coordinates) $r' = r(1 - r^2)$, $\theta' = 25$, $z' = \alpha z$, where α is a constant. Consider separately the three cases $\alpha < 0$, $\alpha = 0$, $\alpha > 0$. Plot orbits in xyz-space for $\alpha = -0.5, 0, 1$.

Van der Pol Systems. Verify that each system in Problems 16–22 satisfies the conditions of Theorem 9.1.1. For each value of μ plot the limit cycle and some orbits that are attracted to it. Estimate the period and the x- and the y-amplitude of each cycle; verify that the larger the value of μ, the longer the period and the larger the y-amplitude of the cycle. The x-amplitude?

16. $x' = y - \mu(x^3 - 10x)$, $y' = -x$; $\mu = 0.1, 2$

17. $x' = y - \mu x(|x| - 1)$, $y' = -x$; $\mu = 0.5, 5, 50$

18. $x' = y - \mu x(x^4 + x^2 - 1)/10$, $y' = -x$; $\mu = 0.1, 1$

19. $x' = y - \mu x(2x^2 - \sin^2 \pi x - 2)$, $y' = -x$; $\mu = 0.5, 5$

20. $x' = y - \mu(x - |x + 1| + |x - 1|)$, $y' = -x$; $\mu = 0.5, 5, 50$

21. $x' = y - \mu(x^3 - x)$, $y' = -x$; $\mu = 0.1, 3, 5, 7, 10$

22. $x' = y - \mu(|x|x^3 - x)$, $y' = -x$; $\mu = 0.5, 1, 3, 5, 7, 10$.

23. *Bifurcation in a van der Pol System* Use a numerical solver to show that as μ decreases from +1 to -1, the equilibrium point of system (7) changes its stability at $\mu = 0$. Describe what happens to the limit cycle as μ sweeps from +1 to -1.

Liénard Equation. The nonlinear, second-order ODE $x'' + f(x)x' + g(x) = 0$, where $f(x)$ and $g(x)$ are continuous and piecewise smooth, is the *Liénard equation* .

Named for the French mathematician and applied physicist, Alfred Liénard (1869–1958)

24. Show that if $y = x' + F(x)$, where $F(x) = \int_0^x f(s)\,ds$, then the Liénard equation can be written in *Liénard system* form as $x' = y - F(x)$, $y' = -g(x)$. Verify that the van der Pol system is a Liénard system. [*Hint*: the xy-plane here is the *Liénard plane*.]

25. Show that $V = y^2/2 + G(x)$, where $G(x) = \int_0^x g(s)\,ds$, is a weak Lypanov function for the Liénard system of Problem 24 if $g(x)F(x)$ is positive for $x \neq 0$. Explain why the origin of the xy-plane is a stable equilibrium point of the Liénard system. [*Hint*: see the SPOTLIGHT ON LYAPUNOV FUNCTIONS.]

26. Plot the orbit of the periodic solution of $x'' + (x^2 - 1)x' + x = 0$ both in the xx'-plane and in the Liénard xy-plane, where y is defined as in Problem 24.

27. The *Rayleigh equation* is $z'' + \mu[(z')^2 - 1]z' + z = 0$. Differentiate the Rayleigh equation with respect to t, then set $x = \sqrt{3}z'$, and show that $x'' + \mu(x^2 - 1)x' + x = 0$. Explain why the ODE in x reduces to a Liénard system if we set $y = x' + \mu(x^3/3 - x)$. Show that the Rayleigh equation has a unique attracting limit cycle for each $\mu > 0$. Plot the cycle and orbits through the points $(1.5, 1.5)$ and $(0.5, 0.5)$ in the zz'-plane for $\mu = 0.1, 1, 5, 10$.

9.2 Solution Behavior in Planar Autonomous Systems

The maximally extended orbits of a planar autonomous system are curves in the state plane. If the rate functions of the system are continuously differentiable, then the orbits completely fill up the plane. By uniqueness no two orbits ever cross or touch, but they may behave in an incredible variety of ways. However, if an orbit is bounded, then, remarkably enough, we can limit our interpretation of its long-term behavior ed to one of only three alternatives. Poincaré[3] saw to the heart of this behavior, and his ideas underlie much of what we have to say in this section.

Cycle-Graphs

Orbits of a damped Hooke's Law spring tend to an equilibrium point as $t \to +\infty$. Orbits of a van der Pol system spiral toward a limit cycle as $t \to +\infty$. In the example below orbits approach a strange hybrid of equilibrium point and cycle that is neither constant nor periodic.

EXAMPLE 9.2.1

Consider the planar autonomous system

$$\begin{aligned} x' &= x(1 - x - 3.75y + 2xy + y^2) \\ y' &= y(-1 + y + 3.75x - 2x^2 - xy) \end{aligned} \tag{1}$$

The x- and y-axes and the line $y = 1 - x$ are composed of orbits. We can show this by replacing y with $1 - x$ on the right sides of the rate equations and observing that $y' = -x'$; we omit the calculations. The axes and the line $y = 1 - x$ intersect at the equilibrium points $(0, 0)$, $(1, 0)$, and $(0, 1)$. Figure 9.2.1 shows the counterclockwise direction of motion along the sides of the orbital triangle that connects these points. Each edge orbit tends to an equilibrium point at a vertex as $t \to +\infty$ and as $t \to -\infty$. The triangle can't be traversed in finite time since it takes infinitely long to trace out each side, so the triangle is *not* a periodic orbit of system (1).

Visual evidence suggests that every orbit inside the triangle spirals outward from the equilibrium point $(0.25, 0.25)$ toward the triangle as t increases (Figure 9.2.1). Figure 9.2.2 shows the component graphs of the spiraling orbit of Figure 9.2.1. The time

Jules Henri Poincaré

[3]History regards the French mathematician Jules Henri Poincaré (1854–1912) as the "last universalist," the last person to understand all the mathematics and much of the physics of his era. He spent most of his professional life at the University of Paris and the École Polytechnique, where he was professor of mathematical physics, probability, celestial mechanics, and astronomy. Every year he lectured on a different subject, his students taking notes that were later published. His research covered most of the areas of the mathematics of his time: groups, number theory, theory of functions, algebraic geometry, algebraic topology (which he developed), mathematical physics, celestial mechanics, partial differential equations, and ordinary differential equations. His first research paper (1878) and his last (1912) were on ODEs. Poincaré and A. M. Lyapunov created the modern approach to ODEs with its emphasis on the general behavior of orbits and solutions, rather than on solution formulas. Poincaré had a talent for good exposition, wrote many popular books on mathematics and science, and was the first (and, so far, the only) mathematician elected to the literary section of the French Institute.

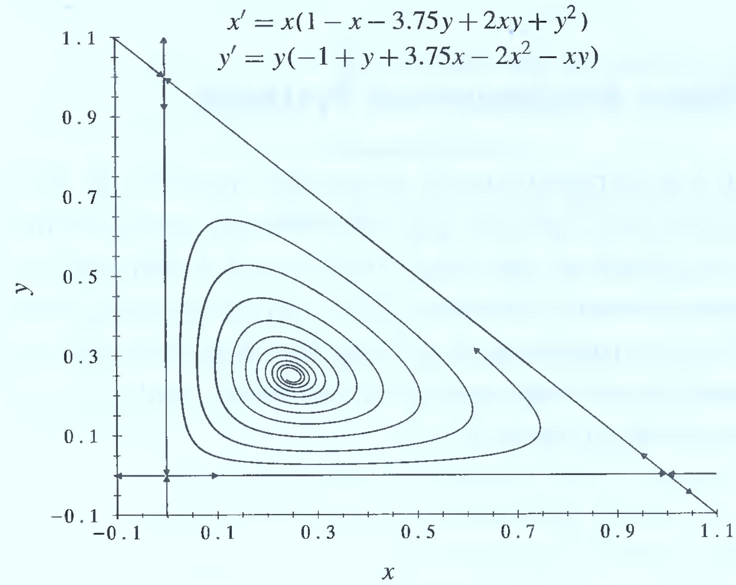


FIGURE 9.2.1 A triangular cycle-graph approached by an outward spiraling orbit (Example 9.2.1). The cycle-graph consists of three vertex equilibrium points and three orbits along the edges.

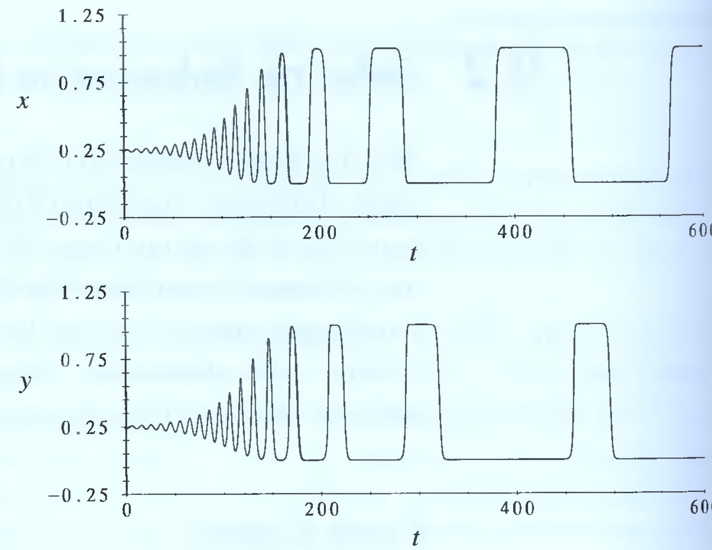
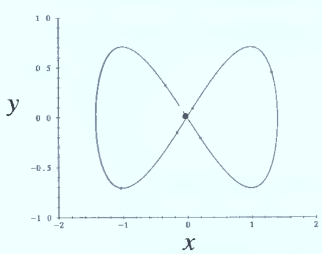


FIGURE 9.2.2 Component graphs of the spiral orbit of Figure 9.2.1: these graphs show that the orbit goes slowly around the corners of the triangle and speeds up near the sides. (Example 9.2.1).

intervals where both component graphs are horizontal correspond to the slow motion near the vertices. The nearly vertical segments of the component graphs correspond to rapid changes in x or y away from the vertices.

The triangle of Figure 9.2.1 is an example of an unusual array of orbits and equilibrium points defined below.

❖ **Cycle-Graph.** A *cycle-graph* (or *polycycle*) of a planar autonomous system is a closed curve in state space that consists of N vertices ($N \geq 1$) and at least N edges. The vertices are equilibrium points of the system and the edges are orbits that tend to vertices as $t \to -\infty$ and as $t \to +\infty$. The graph is coherently oriented along the edges by time's increase (i.e., we can make a round trip around the graph, covering each edge once, by following the arrows).



The triangular cycle-graph of Example 9.2.1 has three vertices and three edges and is oriented counterclockwise. Cycle-graphs may have more edges than vertices as the clockwise-oriented, lazy-eight cycle-graph (one vertex and two edges) in the margin shows (see Problem 22).

Where Do the Orbits Go?

So far in this chapter, orbits that remain bounded as $t \to +\infty$ approach an equilibrium point, a cycle, or a cycle-graph. *There are no other alternatives.* Poincaré and Ivar

Bendixson[4] proved the following fundamental result about long-term behavior:

THEOREM 9.2.1 **Long-Term Behavior in the Plane**

Also known as the Poincaré-Bendixson Theorem.

Suppose that Γ is a maximally extended orbit of the system $x' = f(x, y)$, $y' = g(x, y)$, where f and g are continuous and continuously differentiable. Suppose that as $t \to +\infty$, Γ stays inside a rectangle containing only a finite number of equilibrium points. Then, as $t \to +\infty$, Γ must tend to exactly one of the following:

- an equilibrium point,
- a cycle,
- a cycle-graph

The same alternatives hold as $t \to -\infty$ if Γ stays inside a rectangle as $t \to -\infty$.

A maximally extended orbit is *bounded* if it lies in a box.

These alternatives give us all the possible histories and futures of a bounded and maximally extended orbit Γ. For example, Γ might tend to an equilibrium point as $t \to -\infty$ and to a cycle as $t \to +\infty$, or may tend to a cycle as $t \to -\infty$ and to a cycle-graph as $t \to +\infty$. If Γ is itself an equilibrium point or a cycle, then it tends to itself as $t \to -\infty$ and as $t \to +\infty$. If Γ_1 is a nonconstant orbital edge in a cycle-graph Γ, then Γ_1 approaches an equilibrium point of the cycle-graph as $t \to -\infty$ and an equilibrium point of the cycle-graph (possibly the same point) as $t \to +\infty$. One thing Γ *cannot do* as $t \to +\infty$ is tend to a pair of equilibrium points or to a single nonconstant, nonperiodic orbit. Nor can Γ wander around inside its bounding rectangle without any apparent destination. As time advances (or regresses), Γ must head toward one of the three alternative sets of the theorem. If Γ approaches a cycle or a cycle-graph, it must do so in a spiraling fashion that is consistent with the time orientation of the cycle or cycle-graph.

The positive limit set of the spiraling orbit in Figure 9.2.1 is the triangular cycle-graph, and the negative limit set is the equilibrium point $(0.25, 0.25)$.

The set of points an orbit Γ approaches as $t \to +\infty$ is its *positive limit set* $\omega(\Gamma)$. We define the *negative limit set* $\alpha(\Gamma)$ similarly, except that $t \to -\infty$. The letters α and ω are the first and the last letters of the Greek alphabet, so $\alpha(\Gamma)$ tells us how and where the orbit Γ is born, and $\omega(\Gamma)$ tells how and where it dies.

What can we say about the long-term behavior of Γ if Γ isn't bounded? If, for example, Γ stays in no rectangle as $t \to -\infty$, but remains in some rectangle for all $t \geq t_0$ for some t_0, then the alternatives of Theorem 9.2.1 apply only to $\omega(\Gamma)$. We saw this behavior in Example 9.1.1 where orbits outside the limit cycle $r = 1$ become unbounded as t decreases but spiral toward the limit cycle as $t \to +\infty$. The orbits in Example 9.1.2 reverse this behavior: the orbits born outside the limit cycle $r = 2$ become unbounded as t increases.

The alternatives of Theorem 9.2.1 have profound implications.

[4]The Swedish mathematician Ivar Otto Bendixson (1861–1935) started his career by making important contributions to set theory and point set topology. Later he became intrigued with the qualitative properties of curves generated by planar autonomous systems. Poincaré first proved the alternatives cited in Theorem 9.2.1, but Bendixson gave a more rigorous proof under weaker hypotheses in 1901.

THEOREM 9.2.2 **Unbounded Orbits**

Suppose that the system $x' = f(x, y)$, $y' = g(x, y)$ has no equilibrium points. Then all orbits become unbounded as time increases and decreases.

☞ As always, we assume that f and g are continuously differentiable on $\mathbb{R}^2$.

Proof. Suppose, to the contrary, that an orbit Γ is bounded for all $t \geq t_0$, for some t_0. Then, by Theorem 9.2.1, $\omega(\Gamma)$ must be a cycle because the other two alternatives involve equilibrium points. The interior region of a cycle must contain at least one equilibrium point (see page 527), but this contradicts our assumption that the system has no equilibrium points. This contradiction implies that Γ cannot be bounded as t increases. Similarly, Γ cannot be bounded as t decreases. So Γ becomes unbounded for increasing time and for decreasing time. ■

☞ Without equilibrium points in the system, orbits are born at infinity and return there to die.

THEOREM 9.2.3 **Bounded Orbits**

Every bounded orbit of the system $x' = f(x, y)$, $y' = g(x, y)$ has an equilibrium point or a cycle in its negative limit set and in its positive limit set. If a limit set contains no equilibrium points, then the limit set is a cycle.

Proof. If a limit set is not a cycle, then it must either be an equilibrium point or a cycle-graph, and cycle-graphs contain equilibrium points. ■

To see how a bounded orbit behaves as time tends to $+\infty$ or to $-\infty$, first locate all of the equilibrium points, cycles, and cycle-graphs. Find the equilibrium points by solving the equations $f(x, y) = 0$, $g(x, y) = 0$ simultaneously. Solvers (such as ODE Architect) use Newton's Method to approximate the coordinates of the equilibrium points.

Are There Any Cycles or Cycle-Graphs?

Here is one way to locate a cycle. Suppose that S is a ringlike region without any equilibrium points, and that all orbits intersecting the inner or outer edges of S move into S as time increases. Then there must be at least one cycle in S that also encloses the hole. The cycle must also enclose one or more equilibrium points, but S doesn't have any, so there must be at least one equilibrium point in the hole. The following example illustrates one way to find a ring with the desired properties.

EXAMPLE 9.2.2 **Where Is the Cycle?**

The system

$$\begin{aligned} x' &= y + x\left(1 - 2x^2 - \frac{1}{2}y^2\right) \\ y' &= -x + y\left(1 - 2x^2 - \frac{1}{2}y^2\right) \end{aligned} \tag{2}$$

☞ It takes some algebra to show that there is only one equilibrium point.

has a single equilibrium point at the origin. If there is a cycle, then the cycle must encircle the origin. Let's construct a ring around the origin with the property that orbits which touch its perimeters move into the ring as time increases. Because there are no equilibrium points in the ring itself, every orbit Γ that enters the ring must have a cycle in the ring as its positive limit set.

Put $V = x^2 + y^2$, the square of the distance from an orbital point $x(t)$, $y(t)$ to the origin. Then the derivative of V following the motion is

$$V' = 2xx' + 2yy' = 2(x^2 + y^2)(1 - 2x^2 - y^2/2)$$

V' is positive if $x^2 + y^2 < 1/3$ and V' is negative if $x^2 + y^2 > 3$, because

☞ These inequalities are *not* obvious!

$$1 - 2x^2 - y^2/2 > 1 - 3(x^2 + y^2) > 0 \quad \text{if} \quad x^2 + y^2 < 1/3$$
$$1 - 2x^2 - y^2/2 < 1 - (x^2 + y^2)/3 < 0 \quad \text{if} \quad x^2 + y^2 > 3$$

These rough estimates give us the circular inner and outer perimeters (e.g., $x^2 + y^2 = 0.3$ and $x^2 + y^2 = 3.1$) of a ring with the property that orbits touching either perimeter must move into the ring as time increases (Figure 9.2.3). So, the ring has to have a limit cycle inside. Figure 9.2.3 suggests that this cycle is unique and attracting.

EXAMPLE 9.2.3

Competing Species: No Cycles

The system

$$x' = x(1 - 0.1x - 0.1y), \qquad y' = y(2 - 0.05x - 0.025y) \tag{3}$$

☞ We presented competing species models in Section 7.3.

models the populations of two competing species.The four equilibrium points $(0, 0)$, $(0, 80)$, $(10, 0)$, and $(70, -60)$ of the system lie on the boundary of (or outside) the population quadrant $x \geq 0$, $y \geq 0$, which is why there are no cycles inside the population quadrant and no cycles intersecting the quadrant. The first assertion follows from the fact that a cycle inside the quadrant must enclose an equilibrium point, but there are no equilibrium points inside the quadrant. The second assertion follows because the x- and y-axes are unions of orbits, so orbits can't enter the first quadrant from any other quadrant because they would have to intersect orbits on the axes.

Bendixson discovered a criterion for the *absence* of cycles and cycle-graphs in a simply connected region of the xy-plane. A connected region in the plane is said to be *simply connected* if it has no holes. So we determine that the rectangle described by $|x| \leq 1$, $|y| \leq 2$ is simply connected, but the ring of Example 9.2.2 is not.

THEOREM 9.2.4

Bendixson's Negative Criterion

Suppose that the function $\partial f/\partial x + \partial g/\partial y$ has a fixed sign in a simply connected region R of the xy-plane. Then the system $x' = f(x, y)$, $y' = g(x, y)$ has no cycle or cycle-graph in R.

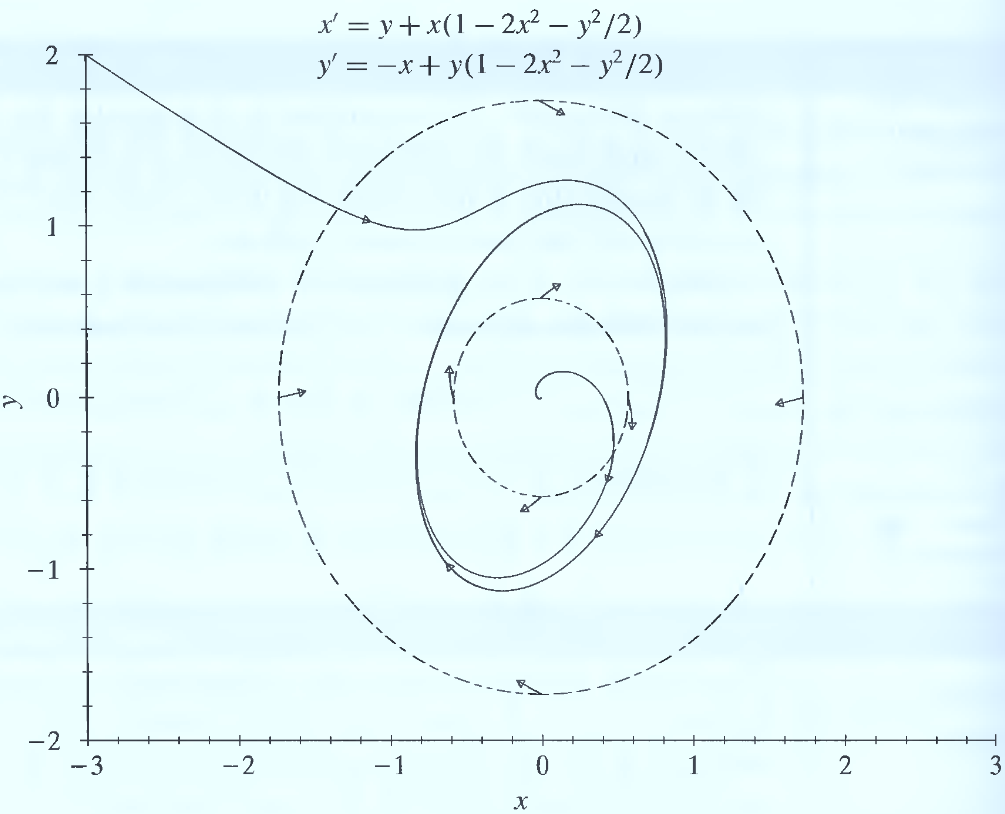


FIGURE 9.2.3 An attracting limit cycle inside a ring (Example 9.2.2).

Proof. Suppose that $\partial f/\partial x + \partial g/\partial y$ is positive in a simply-connected region R and, contrary to the assertion of Theorem 9.2.4, a cycle or cycle-graph Γ lies inside R. Apply Green's Theorem (Theorem B.2.13 in Appendix B.2) to $\partial f/\partial x + \partial g/\partial y$ over the region S consisting of Γ and its interior:

$$0 < \int\!\!\int_S (\partial f/\partial x + \partial g/\partial y)\, dx\, dy \quad [\text{since } \partial f/\partial x + \partial g/\partial y > 0 \text{ in } S]$$

$$= \oint_\Gamma (g\, dx - f\, dy) \qquad [\text{Green's Theorem}]$$

However, $dx = f\, dt$ and $dy = g\, dt$ along Γ, because Γ is an orbit (if it is a cycle) or a union of orbits (if it is a cycle graph) of $x' = f,\ \ y' = g$. This implies that $g\, dx - f\, dy = gf\, dt - fg\, dt = 0$ everywhere on Γ, which is a contradiction:

$$0 < \oint_\Gamma (g\, dx - f\, dy) = 0$$

So there is no cycle or cycle-graph inside the region R where $\partial f/\partial x + \partial g/\partial y$ is positive. A similar proof works if $\partial f/\partial x + \partial g/\partial y$ is negative. ■

Here is a practical application of Bendixson's Negative Criterion.

EXAMPLE 9.2.4

Bendixon's Criterion: Damping Means No Cycles or Cycle-Graphs

ODEs of the form $x'' = g(x, x')$ model the motion of a simple pendulum, a block on the end of a spring, and the current in an RLC circuit. The equivalent system is

$$x' = y, \qquad y' = g(x, y)$$

In this case $f = y$, so the function $\partial f/\partial x + \partial g/\partial y$ reduces to $\partial g/\partial y$. If $\partial g/\partial y$ has a fixed sign in the xy-plane, there can be no cycles or cycle-graphs. For example, the system that models a damped simple pendulum is

$$x' = y, \qquad y' = -a \sin x - by$$

☞ Friction dissipates energy, which is physically why there are no cycles or cycle-graphs.

where a and b are positive constants. The term $-by$ models the effect of friction on the motion of the pendulum, and the function $\partial g/\partial y$ is the negative constant $-b$. So by Theorem 9.2.4 the damped pendulum system has no cycles and no cycle-graphs.

Looking Back and Ahead

The tests and alternatives of this section are qualitative, not quantitative. We described the long-term behavior of the orbits of a planar system in words and in graphs, but not by solution formulas. Results such as these are appropriate at the early stages of an analysis of a complex system that models a physical phenomenon. For example, if the aim is to construct a planar autonomous system with a cycle or a cycle-graph, the system must have an equilibrium point. We use a numerical solver to help analyze orbital structure. Attracting or repelling cycles, cycle-graphs, and equilibrium points are (usually) visible on the screen, and they give useful information about long-term orbital behavior.

We have already described long-term behavior for a single scalar ODE, $x' = f(x)$ by the sign analysis and state-line techniques of Section 2.8. Every nonconstant and bounded orbit on a state line tends to an equilibrium point as $t \to -\infty$ and to another equilibrium point as $t \to +\infty$; there is no room for cycles or cycle-graphs. What happens in three-dimensional state space? That's the subject of Section 9.4. Expect to see unusual behavior of solutions!

PROBLEMS

Limit Sets. Use analytical or graphical techniques to find the positive and the negative limit sets of the orbits through the listed initial points. Sketch some orbits for Problems 3, 4 and 6.

1. $x' = y,\ y' = -x;\ (0, 0), (1, 1)$
2. $x' = y,\ y' = x;\ (1, 1), (1, -1), (1, 0)$
3. *Van der Pol* $x' = y - (x^3/3 - x),\ y' = -x;\ (0.1, 0), (3, 0)$ [*Hint*: see Section 9.1.]
4. *Undamped Simple Pendulum* $x' = y,\ y' = -\sin x;\ (0, 1), (0, \sqrt{2}), (0, 2)$
5. *Polar Form* $r' = r(1 - r^2)(4 - r^2),\ \theta' = 5;\ r_0 = 1/2, 3/2, 5/2;\ \theta_0 = 0$
6. $x' = [x(1 - x^2 - y^2) - y][(x^2 - 1)^2 + y^2],\ y' = [y(1 - x^2 - y^2) + x][(x^2 - 1)^2 + y^2]$; $(1/2, 0), (0, \pm 1), (3/2, 0)$ [*Hint*: the unit circle is a cycle-graph consisting of two equilibrium points joined by two orbital arcs.]

Bendixson Negative Criterion. Show that each ODE has no cycles and no cycle-graphs. [*Hint*: convert each ODE to a system.]

7. *Damped Pendulum* $mLx'' + cLx' + mg\sin x = 0$; m, L, c, and g are positive constants.

8. *Damped Nonlinear Spring* $mx'' + ax' + bx + cx^3 = 0$, where m, a, and b are positive constants and c is any constant.

9. $x'' + (2 + \sin x)x' + g(x) = 0$, where $g(x)$ is any continuously differentiable function.

10. $x'' + f(x)x' + g(x) = 0$; f and g are continuously differentiable and $f(x)$ has a fixed sign.

Bendixson Negative Criterion. Show that each system has no cycle or cycle-graph in the region indicated.

11. $x' = 2x - y + 36x^3 - 15y^2,\ y' = x + 2y + x^2y + y^5$; xy-plane

12. $x' = 12x + 10y + x^2y + y\sin y - x^3,\ y' = x + 14y - xy^2 - y^3$; the disk $x^2 + y^2 \leq 8$

13. $x' = x - xy^2 + y\sin y,\ y' = 3y - x^2y + e^x\sin x$; the interior of the disk $x^2 + y^2 \leq 4$

No Limit Sets. Explain why all orbits of each system are unbounded. [*Hint*: use Theorem 9.2.2.]

14. $x' = x - y + 10, \quad y' = x^2 + y^2 - 1$ **15.** $x' = y, \quad y' = -\sin x - y + 2$

16. $x' = x^2 + 2y^2 - 4, \quad y' = 2x^2 + y^2 - 16$

Any Cycles? Determine whether the following systems have cycles. Find all cycles (graphically or analytically), if any exist. If there are no cycles, explain why not.

17. $x' = e^x + y^2,\ y' = xy$ [*Hint*: does x ever decrease?]

18. $x' = 2x^3y^4 + 5,\ y' = 2ye^x + x^3$ [*Hint*: use Bendixson's Negative Criterion.]

19. *A System in Polar Coordinates* $r' = r\sin(r^2),\ \theta' = 1$

Poincaré–Bendixson Investigation.

20. *Green's Theorem and Averages* Prove that the average value of the function $F(x, y) = \partial f/\partial x + \partial g/\partial y$ on the region R inside a cycle Γ of the system $x' = f(x, y),\ y' = g(x, y)$ must be zero. [*Hint*: the *average value* of F on R is $\int\int_R F(x, y)\,dx\,dy/A$, where A is the area of R. Use Green's Theorem (Theorem B.2.13 in Appendix B.2).]

21. *Contradicting Bendixson?* The quantity $\partial f/\partial x + \partial g/\partial y$ for the system $x' = f = x - 10y - x(x^2 + y^2),\ y' = g = 10x + y - y(x^2 + y^2)$ is $2 - 4x^2 - 4y^2$, which is negative in the region R, $x^2 + y^2 > 0.5$. Show that the unit circle is a cycle that lies entirely in R. Why does this *not* contradict Bendixson's Negative Criterion?

Cycle-Graphs. The following problems explore the nature of cycle-graphs.

22. *Lazy-Eight Cycle-Graph* Show that the system $x' = y + x(1 - x^2)(y^2 - x^2 + x^4/2),\ y' = x - x^3 - y(y^2 - x^2 + x^4/2)$ has equilibrium points at $(0, 0)$ and $(\pm 1, 0)$. Use a numerical solver and plot the orbits through the point $(-3, 2)$ forward in time. Repeat with the orbits through $(\pm 0.5, 2)$, but carry these orbits forward and backward in time. Explain what you see. [*Hint*: look at the graph in the margin on page 530 and at "The Lazy-Eight Cycle Graph" file in the "Golden ODEs" folder in the ODE Architect Library.]

23. *Triangle Cycle-Graph* System (1) has a triangular cycle-graph that is the positive limit set of orbits inside the triangle. Let's explore the regions *outside* the triangle.

(a) Plot a comprehensive portrait of the orbits of the system in the region $|x| \leq 2,\ |y| \leq 2$. Mark the six equilibrium points in that region. On the basis of what the nearby orbital arcs look like, label each point as stable or unstable, and name each as a node (stable, unstable?), a saddle, or a spiral point (stable, unstable?).

(b) Use the Jacobian matrix technique of Section 8.2 to verify that the equilibrium points $(0, 0)$, $(1/4, 1/4)$, $(4/3, 4/3)$ have the stability characteristics claimed in **(a)**.

(c) *Cycle-Graph Sensitivity: Bifurcation* Replace the coefficient 3.75 in the first equation of system (1) by the parameter c, but leave the second equation as is. For $c = 4$ and 3.5, plot portraits of the orbits. On the basis of your graphs, what do you think happens to the orbits as c increases through the critical value $c = 3.75$?

24. *Dulac's Criterion for No Cycles* Suppose that R is a simply connected region in the plane. Suppose that f, g, and K are continuously differentiable in R. Suppose that $\partial(Kf)/\partial x + \partial(Kg)/\partial y$ has a fixed sign in R. Show that the system $x' = f,\ y' = g$ can't have a cycle or a cycle-graph in R. [*Hint*: read the proof of Bendixson's Negative Criterion. Then adapt the proof to $\partial(Kf)/\partial x + \partial(Kg)/\partial y$.]

Modeling Problems.

25. *Competing Species* Explain the meaning of each term in the rate functions using the vocabulary of a model for competing species: $x' = x(1 - 0.1x - 0.1y)$, $y' = y(2 - 0.05x - 0.025y)$. Plot orbits in the population quadrant, and then make a complete analysis of the ultimate fate of each species in terms of the initial values of your orbits.

26. *Ragozin's Negative Criterion for Interacting Species* Suppose that a two-species interaction is modeled by the system $x' = xF(x, y),\ \ y' = yG(x, y)$. The two species are *self-regulating* if $\partial F/\partial x$ and $\partial G/\partial y$ are each negative throughout the population quadrant. If the x-species, say, is self-regulating, then the negativity of $\partial F/\partial x$ implies that the per unit growth rate F diminishes as x increases. Use Dulac's Criterion (Problem 24) to show that the system of ODEs of a pair of self-regulating species has no cycles in the population quadrant. [*Hint*: let $K = (xy)^{-1}$ for $x > 0,\ \ y > 0$.]

27. Use ODEs to draw the face of a cat.

Scaling Time.

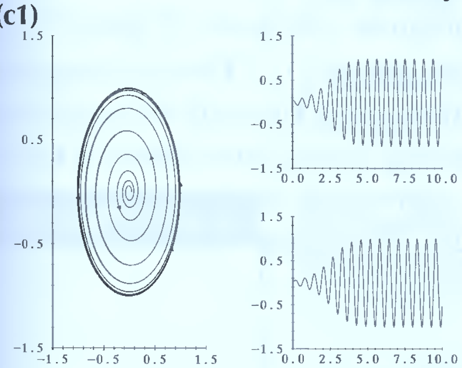
28. Suppose that $x(t)$, $y(t)$ is a solution of the system $x' = f(x, y),\ \ y' = g(x, y)$. Rescale the time variable from t to s by setting $dt/ds = h(x(t), y(t))$ for a given function $h(x, y)$ that has a fixed sign. The scaled system has the form $dx/ds = fh,\ dy/ds = gh$.

(a) If $h(x, y)$ is positive everywhere, explain why the two systems have identical orbits with the same orientation induced by time's advance. What if $h(x, y)$ is everywhere negative?

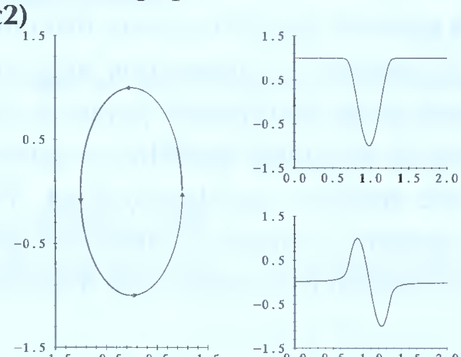
(b) Suppose that $h(x, y)$ is positive everywhere except that h is zero at a point p that lies on a nonconstant orbit Γ of the unscaled system. What happens to the corresponding orbit of the scaled system?

(c) Suppose that $f = x - 10y - x(x^2 + y^2)$ and $g = 10x + y - y(x^2 + y^2)$, while the factor $h = 1 - \exp[-10(x-1)^2 - 10y^2]$. Explain why the unscaled system has a limit cycle (graphs **(c1)** and **(c2)**) and the scaled system has a cycle-graph (with one vertex and one edge) on the unit circle. Plot the orbit through the point $(0.5, 0)$ and the corresponding component curves for each system (graphs **(c1)** and **(c3)**). Do the cycle and the cycle-graph attract? Explain what you see in these graphs.

(c1)



(c2)



(c3)