Traveling Waves in Nonlinear Reaction–Diffusion Equations

Consider a reaction–diffusion equation with a nonlinear source term:

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} + f(c), \tag{7.76}$$

where $f(c)$ is the cubic polynomial $f(c) = Ac(1-c)(c-\alpha)$, with $0 < \alpha < \frac{1}{2}$. While real chemical reactions are not modeled exactly by a cubic polynomial, the reaction term has features that resemble those of several more realistic reactions, and so is worthy of our attention. This equation can be (and has been) used to understand features of action potential propagation in nerve axons, calcium fertilization waves in frog eggs, and cyclic AMP waves in slime molds.

A key feature of this reaction term is that it has three zeros (0, α, and 1), two of which (0 and 1) are stable. Linear stability is determined by the sign of $f'(c)$ at the rest point, and if $f'(c_0) < 0$, the rest point c_0 is stable. In this problem, however, there is a stronger type of stability in that the solution of the ordinary differential equation $\frac{dc}{dt} = f(c)$ approaches either $c = 0$ or $c = 1$ starting from any initial position except $c = \alpha$.

The function $f(c)$ can be thought of as a switch. If c is somehow pushed slightly away from 0, it returns quickly to 0. However, if c is pushed away from 0 and exceeds α, then it goes to 1. Thus, the level α is a threshold for c. Because it has two stable rest points, equation (7.76) is often called the bistable equation.

Traveling Wave Solutions

An interesting and important problem is to determine the behavior of the bistable equation when a portion of the region is initially above the threshold α and the remainder is initially at zero. To get some idea of what to expect it is useful to perform a numerical simulation. For this numerical simulation we use the method of lines to solve the differential equations

$$\frac{dc_0}{dt} = \frac{D}{\Delta x^2}\left(c_1(t) - c_0(t)\right) + f(c_0), \tag{7.77}$$

$$\frac{dc_n}{dt} = \frac{D}{\Delta x^2}\left(c_{n+1}(t) - 2c_n(t) + c_{n-1}(t)\right) + f(c_n), \qquad n = 1, 2, \ldots, N-1, \tag{7.78}$$

$$\frac{dc_N}{dt} = \frac{D}{\Delta x^2}\left(c_{N-1}(t) - c_N(t)\right) + f(c_N). \tag{7.79}$$

The simulation shows that the variable c quickly changes into a profile that is a transition between $c = 0$ on the bottom and $c = 1$ on the top (Figure 7.8). After this transitional profile is formed, it moves without change of shape from top to bottom at (what appears to be) a constant velocity.

This numerical solution suggests that we should try to find a translationally invariant solution. A translationally invariant solution is one that does not change its value along any straight line $x + st = x_0$, for an appropriately chosen value of s, the wave speed. Thus, we look for special solutions of the bistable equation of the form

$$c(x, t) = U(x + st), \tag{7.80}$$

with the additional property that $\lim_{\xi\to-\infty} U(\xi) = 0$, $\lim_{\xi\to\infty} U(\xi) = 1$. Notice that since

$$\frac{\partial c(x,t)}{\partial x} = \frac{d}{d\xi}U(\xi)\frac{\partial \xi}{\partial x} = \frac{d}{d\xi}U(\xi) \text{ and } \qquad \frac{\partial c(x,t)}{\partial t} = \frac{d}{d\xi}U(\xi)\frac{\partial \xi}{\partial t} = s\frac{d}{d\xi}U(\xi), \tag{7.81}$$

in this translating coordinate system, the bistable equation becomes the ordinary differential equation

$$s\frac{dU}{d\xi} = D\frac{d^2U}{d\xi^2} + f(U). \tag{7.82}$$

There are two ways to try to solve (7.82). An exact solution can be found in the special case that f is a cubic polynomial. There are several other examples of functions f for which exact solutions can be found, but this method does not work in most cases. A more general method is to examine (7.82) in the phase plane, which we will do below.

The exact solution can be found for the cubic polynomial f as follows. Since we want $\lim_{\xi\to-\infty} U(\xi) = 0$, $\lim_{\xi\to\infty} U(\xi) = 1$, we guess a relationship between $dU/d\xi$ and U of the form

$$\frac{dU}{d\xi} = aU(1 - U), \tag{7.83}$$

for some positive number a. It follows that

$$\frac{d^2U}{d\xi^2} = a(1 - 2U)\frac{dU}{d\xi}. \tag{7.84}$$

Substituting this into (7.82) and factoring out $U(1 - U)$, we find that

$$as = a^2D(1 - 2U) + A(U - \alpha). \tag{7.85}$$

This identity holds for all U only if

$$a^2 = \frac{A}{2D}, \qquad s = \sqrt{AD/2} \cdot (1 - 2\alpha). \tag{7.86}$$

The solution is found by quadrature from (7.83) to be

$$U(\xi) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{1}{2}\sqrt{\frac{A}{2D}}\xi\right). \tag{7.87}$$

The analysis used for finding traveling waves using phase portraits works for any bistable function f. We begin by writing the traveling wave (7.82) as the two-dimensional system

$$\frac{dU}{d\xi} = W, \tag{7.88}$$

$$\frac{dW}{d\xi} = sW - f(U). \tag{7.89}$$

This system has three critical points, at $(U,W) = (0,0), (\alpha,0)$, and (1,0). The linearized stability of these critical points is determined by the roots of the characteristic equation

$$\lambda^2 - s\lambda + f'(U_0) = 0, \tag{7.90}$$

where U_0 is any one of the three steady rest values of U.

If $f'(U_0)$ is negative, then the critical point $(U_0, 0)$ is a saddle point. To find a traveling wave solution, we seek a trajectory that leaves the saddle point at $(U,W) = (0,0)$ and ends up at the saddle point at $(U,W) = (1,0)$. We can implement this (almost) numerically. If we start with an initial point close to the origin along the straight line $W = \lambda U$ in the positive quadrant, with λ the positive root of the characteristic equation $\lambda^2 - s\lambda + f'(0) = 0$, and integrate for a while, one of two things will occur. If s is relatively small, the trajectory will cross the U–axis before reaching $U = 1$, while if s is relatively large, the trajectory will increase beyond $U = 1$ and become quite large. By adjusting the parameter s one can find trajectories that barely miss hitting the point $(U,W) = (1,0)$ by crossing the U–axis or by exceeding $U = 1$ and becoming large (Figure 7.9). A trajectory that comes close to the saddle point at $(U,W) = (1,0)$ is a numerical approximation to the traveling wave solution, and the value of s for which this nearly connecting trajectory is attained is a good approximation for the wave speed.

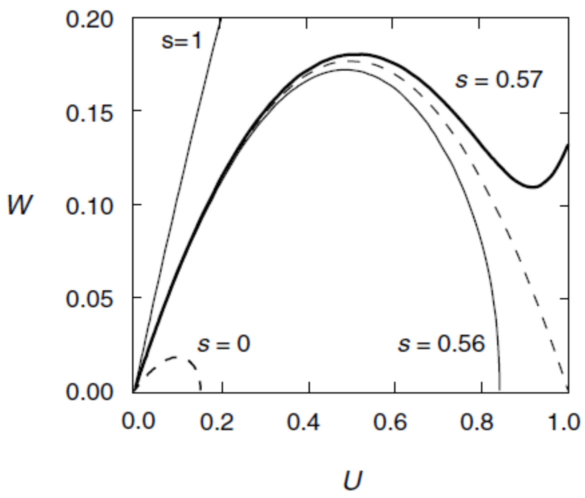

Figure 7.9 Phase plane portrait of possible traveling wave trajectories.