Periodic linear systems Floquet theory.

$\dot{x} = A(t)x$; $A(t+T) = A(t)$, $T>0$

A- periodic function.

$x(t+T)$ is also solution if $x(t)$ is.

The main result for periodic linear system is:

Theorem 3.15. (Floquet)

The principal matrix solution to the periodic linear system $\dot{x} = A(t)$, $A(t+T) = A(t)$ has the form:

$$\Pi(t, t_0) = P(t, t_0) \exp\{(t-t_0) Q(t_0)\}$$

where $Q_0(t_0)$ is a constant matrix.

and $P(t+T, t_0) = P(t, t_0)$; $P(t_0, t_0) = I$.

$Q_0(t_0)$ and its eigenvalues are defined not uniquely and can be complex. Matrix $P(t, t_0)$ can also be complex.

It makes it possible to describe stability of solutions similarly to the case of constant matrix A, because term $P(t, t_0)$ is periodic and bounded

Lemma 3.14. Principal matrix solution of a periodic system satisfies the relation (periodic with respect to pair (t, t_0))

$$\Pi(t+T, t_0+T) = \Pi(t, t_0)$$

This relation follows from the observation that $\Pi(t, t_0)$ maps initial data $x(t_0) = x_0$ to the corresponding solution $x(t)$ of the equation $\dot{x} = A(t)x$ with periodic $A(t+T) = A(t)$ at time t.
If we solve this equation with initial data $x(t_0+T) = x_0$ instead, we get exactly the same solution "shifted" in time because the matrix $A(t)$ is periodic (repeats itself after T) and solutions are unique.
It implies that $\Pi(t+T, t_0+T) = \Pi(t, t_0)$

We can express this argument by

$$(\Pi(t+T, t_0+T))' = A(t+T)\,\Pi(t+T, t_0+T) =$$
$$= A(t)\,\Pi(t+T, t_0+T),$$

$\Pi(t_0+T, t_0+T) = I$; $\Rightarrow \Pi(t+T, t_0+T)$ solves the equation $\Pi' = A(t)\Pi$ and satisfies $\Pi(t_0+T, t_0+T) = I$ $\Rightarrow$
it must be equal $\Pi(t, t_0)$ by uniqueness. and by definition of $\Pi(t, t_0)$

The idea of the proof to the Theorem 3.15 by Floquet is based on a Poincaré map in time variable. Consider a mono-dromy matrix

$M(t_0) = \Pi(t_0+T, t_0)$ - describing a shift of the solution on T in time.

By the lemma 3.14.

$$M(t_0+T) = M(t_0)$$

Application of this operator l-times is equivalent to applying the principal matrix solution on the interval lT:

$$\Pi(t_0+lT, t_0) = \Pi(t_0+lT, t_0+(l-1)T) \times$$
$$\times \Pi(t_0+(l-1)T, t_0+(l-2)T) \dots \Pi(t_0+T, t_0) =$$
$$= (M(t_0))^l.$$

We see that shift in time in $l \cdot T$ corresponds to the power l of the monodromy matrix $M(t_0)$.

We live to use this idea to express a general shift in time as an exponent for an arbitrary time t.

We live to find a matrix $Q(t_0)$ such that

$$M(t_0) = \exp(T \cdot Q(t_0))$$

It would give possibility to calculate

$$\boxed{(M(t_0))^{t/T} \text{ as } = \exp(t \cdot Q(t_0))}$$

Formally we can write that we express $Q(t_0)$ as

$$Q(t_0) = \ln(M(t_0)) \frac{1}{T}$$

but we need to give a sense to this expression. In the same way as we did for exp(A), we use power series for logaritm

$$\ln(1+x) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1} x^j}{j}, \quad |x|<1$$

to define $\ln(A) = B$ for $A = \exp(B)$)
We can consider only the case when A is a Jordan canonical matrix and even

$\underbrace{A = \alpha I + N}_{\text{Jordan block}}$ consider only one block:

Because if $A = U J U^{-1}$ with J - Jordan canonical matrix,

$\ln(A) = U \ln(J) U^{-1}$ in the same way as it is shown for $\exp(A)$ or for any other function with convergent power series.

$$\ln\Bigg(\underbrace{\begin{bmatrix} \alpha & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \alpha \end{bmatrix}}_{n}\Bigg) = \ln(\alpha) I + \sum_{j=1}^{n-1} \frac{(-1)^{j+1}}{j} N^j$$

(higher powers of N are all zero)

$\boxed{\text{Lemma 3.34}}$ $\rightarrow$ It shows that $\ln(A)$ can be defined if and only if all eigenvalues of A are nonzero ($\det A \neq 0$).

If some of eigenvalues are negative or complex, $\ln(A)$ will be complex because $\ln(z) = \ln|z| + i\,\mathrm{agr}(z)$ for a complex number z. Argument of a complex number is defined only up to $2\pi i$ (we can add a complete angle without changing the position of the number on the complex plane). Therefore $\ln(A)$ is not unique either.

For real $\alpha < 0$ $\quad \ln(\alpha) = \ln(|\alpha|) + i\pi$

$\det(M(t_0)) \neq 0$ because it is a value of principal matrix solution at a particular time $M(t_0) = \Pi(t_0+T, t_0)$ and the last one is not degenerate. (it follows also from Liouvilles formula)
$$\det(M(t_0)) = \exp\left(\int_{t_0}^{t_0+T} tr(A(s))\,ds\right).$$

It finishes argumentation for the expression $Q(t_0) = \ln(M(t_0))\frac{1}{T}$.

Eigenvalues (ρ_i) of the monodromy matrix $M(t_0)$ are called Floquet multipliers.

Eigenvalues (γ_i) of $Q(t_0)$ are called Floquet exponents. $\rho_i = e^{T\gamma_i}$. Taking $M(t_1) = \Pi(t_1+T, t_1) = D$

Eigenvalues ρ_i of $M(t_0)$ are independent of t_0 because $M(t_0)$ and $M(t_1)$ are similar:
$$M(t_1) = \Pi(t_1+T, t_0+T)M(t_0)\Pi(t_0, t_1) = \Pi(t_1, t_0)M(t_0)\Pi(t_1, t_0)^{-1}$$

Finally we show that defining

$$\boxed{P(t,t_0) = \Pi(t,t_0)\, e^{-Q(t_0)(t-t_0)}}$$

(5a)

we observe that

$$\Pi(t,t_0) = P(t,t_0)\, e^{Q(t_0)(t-t_0)}$$

and we check periodicity of $P(t,t_0)$

$$P(t+T,t_0) = \Pi(t+T,t_0)\, e^{-Q(t_0)(T+t-t_0)} =$$

$$= \left(\Pi(t+T,t_0)\, e^{-Q(t_0)}\right) e^{-Q(t_0)(t-t_0)} =$$

$$= \Pi(t,t_0)\, e^{-Q(t_0)(t-t_0)} = P(t,t_0)$$

It finishes proof of the corollary. $\left(P(t+T,t_0) = P(t,t_0)\right)$

$\blacksquare$

Stability of solutions to linear periodic systems.

Corollary Periodic linear system is stable if it has all Floquet multipliers $|\rho_j| \leq 1$ (Floquet exponents $\operatorname{Re}\gamma_j \leq 0$) and for those ρ_j with $|\rho_j| = 1$ (correspondingly $\operatorname{Re}\gamma_j = 0$) the algebraic and the geometric multiplicities of these eigenvalues must be equal (corresponding Jordan blocks diagonal)

A Periodic linear system is asymptotically stable if all Flouquet multipliers $|\rho_j| < 1$ (Floquet exponents have $\operatorname{Re}\gamma_j < 0$)

Proof. It follows from the observation that $\Omega(t, t_0) = P(t, t_0)\, e^{Q(t_0)(t-t_0)}$

where $P(t, t_0)$ is periodic continuous and therefore bounded matrix. Therefore boundedness of $\Omega(t, t_0)$ for all t and property $\|\Omega(t, t_0)\| \underset{t\to\infty}{\longrightarrow} 0$ is governed only by properties of eigenvalues of $Q(t_0)$. Requirements in the corollary just repeat requirements

for stability and asymptotic stability of the linear system 7.
$\dot{x} = Q(t_0)x$ with constant matrix $Q(t_0)$
If this system is stable (asymptotically stable) then the original periodic system is stable (asymptotically stable).