General linear non-autonomous systems.

$\dot{x} = A(t)x$, $x(t_0) = x_0$, $A \in C(I, R^{n\times n})$ - a matrix - continuous in t on an interval I. $A(t)x$ is always Lipschitz

Theorem 3.9. The system has a unique solution satisfying initial condition $x(t_0) = x_0$, on the whole interval $t \in I$, $t_0 \in I$.

It follows directly from Th. 2.17 on equations $\dot{x} = f(x,t)$ with $f(t,x)$ rising not faster than linearly in x.

An independent proof. Let $C = \max_{t \in [0,T]} \|A(t)\|$

Let $\varphi(t)$ = solution and $\|\varphi(t)\| = r(t)$

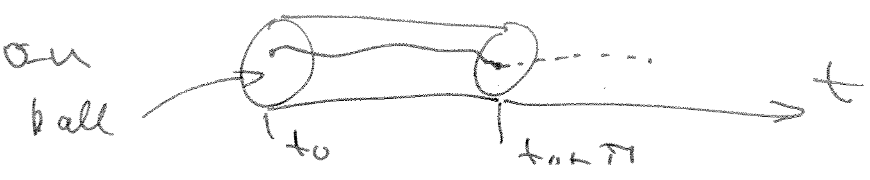
If $L(\tau) = \ln(r^2)$ $\quad t_0 < t < T$

$$\dot{L} \le \frac{2r\dot{r}}{r^2} \le 2C$$

$$L(t) \le L(t_0) + 2C(t - t_0)$$

$$\Rightarrow \|\varphi(t)\| \le e^{C(t-t_0)} \|\varphi(t_0)\|$$

It means that solution $\varphi(t)$ does non leave a Ball with radius $\|e^{C(T-t_0)}\|\varphi(t_0)\|$ and must have an extension



Superposition principle.

A sum of solutions to a linear system $\dot{x} = A(t)x$ is also a solution.
Solutions to this equation form a vector space.

Theorem 3.10 The solutions to $\dot{x} = A(t)x$ form an n-dimensional vector space.
There is a matrix-valued solution $\Pi(t, t_0)$ such that the solution satisfying initial data $x(t_0) = x_0$ is given by

$$x(t) = \Pi(t, t_0)x_0 \ ; \quad \dot{\Pi}(t, t_0) = A(t)\,\Pi(t, t$$

$\Pi(t, t_0)$ is called principal matrix solution. $\Pi(t_0, t_0) = \mathbb{I}$ - unit matrix.

Proof. Suppose there are n solutions $x_1(t), x_2(t) \dots, x_n(t)$ that are linearly independent at $t_0 = t$ and are linearly dependent at some $t = t_*$.
Then $\forall \{c_i\}_{i=1}^{n}$ such that $c_i \neq 0$ all together, $\sum_{i=1}^{n} c_i\, x_i(t_0) \neq 0$
and $\sum_{i=1}^{n} c_i\, x_i(t_*) = 0$ (linear dependent)

We observe that $X(t) - \sum_{i=1}^{n} c_i\, x_i(t)$ is the solution to

$\dot{x} = A(t)x$, because of the superposition principle and
$X(t_0) \neq 0$; $X(t_*) = 0$.

The last is impossible, therefore $X_1(t) \dots x_n(t)$ must be linearly independent for all $t \in I$ if they are linearly independent at time t_0. It proves that the space of solutions is n-dimensional, because the space of initial data $x(t_0) = x_0$ is n-dimensional. ▨

The second claim in the theorem follows by the explicit construction and the superposition principle.
We solve the equation $\dot{x} = A(t)x$ with initial data δ_j; $j = 1, \dots n$, where δ_j - standard basis vectors in R^n $(1, \dots 0)^T$, $(010 \dots 0)^T$, $(0010 \dots 0)^T$ etc. that are columns in the unit matrix. $\mathbb{I}$.
Corresponding solutions $\phi(t, t_0, \delta_j)$ put in this order in a matrix

$$\Pi(t, t_0) = [\phi(t, t_0, \delta_1), \dots, \phi(t, t_0, \delta_n)]$$

give the principal matrix solution
Any vector $x_0 = \mathbb{I} x_0$ and
$\phi(t, t_0, x_0) = \Pi(t, t_0) x_0$ for an arbitrary initial data at time t_0.

$\Pi(t, t_0)$ is an exponent $\exp(At)$ in the case when $A(t)$ is independent of time (constant matrix.)

In the general case there is no explicit expression for the principal matrix solution $\Pi(t, t_0)$.

The matrix $\Pi(t, t_0)$ is non degenerate because it's non-degenerate for $t = t_0$ (unit matrix $\Pi(t_0, t_0) = \mathbb{I}$) and its column solutions are linearly independent for all other times because of the proof above.

It makes that $\Pi(t, t_0)$ is a one to one mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ and again proves that the set of all solutions is an n-dimensional vector space.

Example. $\begin{cases} x_1' = x_1 + t x_2 \\ x_2' = 2x_2 \end{cases}$

find $\Pi(t, t_0)$. $\Pi(t, t_0)$ has the first column - solution with $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and the second column - with $x_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ gives $x_2 \equiv 0$; $x_1 = e^{t - t_0}$

$x_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ gives $x_2 = e^{2(t - t_0)}$; $x_1 = e^{2(t-t_0)}(t-1) - e^{(t-t_0)}(t_0 - 1)$

Therefore $\Pi(t, t_0) = \left(\begin{array}{c|c} e^{t-t_0} & e^{2(t-t_0)}(t-1) - e^{t-t_0}(t_0-1 \\ \hline 0 & e^{2(t-t_0)} \end{array}\right.$

We observe that $t_0 \leq t_1 \leq t$ 4

$$\Pi(t, t_1)\,\Pi(t_1, t_0) = \Pi(t, t_0)$$

because both expressions solve the equation $\dot{\Pi}(t, t_0) = A(t)\,\Pi(t)$ and they coincide at the point $t = t_1$, therefore must be equal everywhere by uniqueness of solutions. The meaning of the equation is simple: the system moves from time t to t_1 and then from t_1 to t and it is the same as to move all the way from t_0 to t, because of the uniqueness.

<u>Definition</u>

<u>Fundamental matrix solution</u> is an arbitrary matrix solution such that $\det(\bar{U}(t)) \neq 0$; $\dot{U} = A(t)U$

It is easy to observe that for arbitrary fundamental matrix solution $U(t)$

$$U(t)\left(U(t_0)\right)^{-1} = \Pi(t, t_0),$$ because

$$\bar{U}(t_0)\left(\bar{U}(t_0)\right)^{-1} = I.$$

It is easy to observe that

$$\left(\Pi(t, t_0)\right)^{-1} = \Pi(t_0, t)$$

Lemma 3.11 for any fundamental matrix solution

$$\det(U(t)) = \det(U(t_0)) \exp\left\{ \int_{t_0}^{t} tr(A(s))\, ds \right\}$$

$$tr A = \sum_{i=1}^{n} A_{ii}.$$

Direct proof. Let $A = a_{jk}(t)$; $U(t) = y_k^j(t)$

$\Delta(t) = \det U(t)$

$$\dot{\Delta}(t) = \sum_{j=1}^{n} \det(U_j(t)) \quad \text{where } U_j(t)$$

is a matrix taken from $U(t)$ by exchanging its j-th row $(y_1^j, \dots y_n^j)$ by derivatives $\left((y_1^j)', \dots (y_n^j)'\right)$.

U is a solution to $U' = A(t)U$, therefore $\left(y_k^j(t)\right)' = \sum_{i=1}^{n} a_{ji} y_k^i$ —

j-th row in U_j consists of j-th row in $U(t)$ times a_{jj} and a linear combination of other rows in U_j =>

$\det U_j = a_{jj} \det(U)$; =>

$\Delta'(t) = (tr A) \cdot \Delta(t)$ => $\Delta(t) = \Delta(t_0) e^{\int_{t_0}^{t} tr A\, ds}$ ∎

Inhomogeneous linear systems.

<u>Theorem 3.12</u> Solution to inhomogeneous system with initial condition

$$\dot{x} = A(t) + g(t), \quad x(t_0) = x_0$$

is $$x(t) = \Pi(t, t_0)\, x_0 + \int_{t_0}^{t} \Pi(t, s)\, g(s)\, ds.$$

(Variant of Duhamel's formula)

<u>Proof.</u> Check by direct calculation and substituting to <u>the equation.</u>

One can guess it by variation of parameter. Let

$$x(t) = \Pi(t, t_0)\; c(t)\,; \quad c(t_0) = x_0.$$

$$\dot{x} = A(t)\, x(t) + \Pi(t, t_0)\, \dot{c}(t)$$

$$\Rightarrow \dot{c}(t) = \Pi(t_0, t)\, g(t)$$

$$c(t) = x_0 + \int_{t_0}^{t} \Pi(t_0, s)\, g(s)\, ds$$

$$\Rightarrow x(t) = \Pi(t, t_0)\, x_0 + \int_{t_0}^{t} \Pi(t, s)\, g(s)\, ds$$

$$\Pi(t, t_0)\, \Pi(t_0, s) = \Pi(t, s)\ !$$