Lecture notes on linear autonomous systems of ODE

IVP: $\dot{x}(t) = A \cdot x(t)$; $x(0) = x_0$; $x(t), x_0 \in \mathbb{R}^n$ (or $\mathbb{C}^n$)

A – $n \times n$ matrix

Notations for scalar product and norm

$x \cdot y \triangleq \sum_{j=1}^{n} x_j^* \cdot y_j$, x^* - complex conjugate

$\|x\|^2 = x \cdot x$. $A^k = A \cdot A^{k-1}$; $A^0 = I$ - unit matrix

Reformulate I.V.P. as integral equation

$$x = \int_0^t A x(s) ds + x_0$$

and try to solve it by Picard iterations:

$$x_1 = \int_0^t A x_0 ds + x_0 = x_0 + t A x_0$$

$$x_2 = \int_0^t A x_1(s) ds + x_0 = x_0 + t A x_0 + \frac{t^2}{2} A^2 x_0$$

$$\vdots$$

$$x_{k+1} = \int_0^t A x_k(s) ds + x_0 = \sum_{j=0}^{k+1} \frac{t^j}{j!} A^j x_0$$

We whant to approximate solution to IVP. $x(t)$ by the limit of $\lim_{k \to \infty} x_k(t)$ if it exist.

Proo of convergence is based on several statements formulated as Problems.

$n \times n$ Matrices with norm $\|A\| = \sup_{\|x\|=1} \|Ax\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$, $x \in \mathbb{R}^n$ (or $x \in \mathbb{C}^n$ - complex vector space) is a complete vector space (Banach space).

Axioms ov vector space is easy to check.

For $\mathbb{C}^n$ $\|x\| = (x \cdot x)$; $x \cdot x = \sum_{j=1}^{n} x_j^* \cdot x_j$

Competeness of the matrix space is formulated in problem 3.1. (2

Proof. Let $\{A_i\}_{i=1}^{\infty}$ be Cauchy sequence: $\forall \varepsilon > 0$ $\exists N_\varepsilon > 0$: $\|A_i - A_k\| \leq \varepsilon$ if $i, k > N_\varepsilon$.

Let $x \neq 0$ be arbitrary in $\mathbb{C}^n$. Consider a sequence $x_i \in \mathbb{C}^n$: $x_i = A_i x$. We will show that $\{x_i\}$ is a Cauchy sequence in $\mathbb{C}^n$.

$\|x_m - x_k\| \leq \|(A_m - A_k)x\| \leq \|A_m - A_k\| \|x\| \leq$
$\leq \varepsilon \|x\|$. $\|x\|$ is fixed $\Rightarrow \{x_i\}$ is a Cauchy sequence. $\mathbb{C}^n$ is a complete space $\Rightarrow \exists \lim_{i \to \infty} x_i = y \in \mathbb{C}^n$.

Observe that $\|x_m - y\| \leq \varepsilon \|x\|$, $m > N_\varepsilon$ by going to the limit in the estimate for $\|x_m - x_k\|$ with $k \to \infty$.

y is a linear function of x because limit is linear and $y = \lim_{k \to \infty} A x_k$; $x = \lim_{k \to \infty} x_k$

Therefore there is a matrix A such that it represents $y = Ax$. Therefore for $k > N_\varepsilon$

$$\|A_k - A\| = \sup_{\|x\| \neq 0} \frac{\|A_k x - Ax\|}{\|x\|} =$$

$$\sup_{\|x\| \neq 0} \frac{\|x_k - y\|}{\|x\|} \xrightarrow[k \to \infty]{} 0$$, becaus $x_k \xrightarrow[k \to \infty]{} y$

Therefore $\lim_{k \to \infty} A_k = A$ and the space of matrices with norm $\|A\|$ is complete (Cauchy sequences converge) (Banach space)

Problem 3.2 states that $\|AB\| \le \|A\| \cdot \|B\|$ is proved by using the definition of the norm twice. and implies similarly as in analysis in $\mathbb{R}$ that if $A_j \to A$, $B_j \to B$ then $A_j \cdot B_j \to AB$

These properties imply that series of matrices in $\mathbb{C}^{n\times n}$: $\sum_{j=1}^{\infty} A_j$ with convergen series of norms : $\sum_{j=1}^{\infty} \|A_j\| < \infty$ must converge itself (Problem 3.4) The proof uses completeness of the matrix space and is similar to classical Weierstrass criterion for series of functions.

These statements imply that

$$\exp(At) = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$$ converges because series $\sum_{k=0}^{\infty} \frac{t^k \|A\|^k}{k!}$

converges. uniformly on any finite time interval. It lets us go to the limit with $k \to \infty$ in the expression for Picard iterations:

$$x_{k+1}(t) = \int_0^t A\, x_k(s)\, ds + x_0$$

and conclude that the limit $\lim_{k\to\infty} x_k(t) = x(t)$ satisfies the integral equation

$$x(t) = \int_0^t A\, x(s)\, ds + x_0.$$ and therefore by differentiating it $\dot{x} = Ax$; $x(0) = x_0$

So the exponential $\exp(At)x_0 = x(t)$ gives solution to the initial value problem $\dot{x}(t) = Ax(t)$; $x(0) = x_0$

This solution is also unique, but we leave this question to a moment when we consider it in more general form.

Lemma 3.1. Suppose A and B commute. ($AB = BA$) then $\exp(A+B) = \exp(A)\exp(B)$

The proof is similar to one in the one-dimensional case: multiplying series for $\exp(A)$ and $\exp(B)$ and using Newton's binom together with $AB = BA$ for these particular matrices leads to a series for $\exp(A+B)$.

Linear change of variables.

Linear change of variables $x = Uy$; $y = U^{-1}x$ with an invertible matrix U gives that $\exp(A) = U\exp(U^{-1}AU) \cdot U^{-1}$. and $\dot{y}(t) = (U^{-1}AU)y$ with $y(0) = U^{-1}x_0$. It follows from the relation $A^k = U(U^{-1}AU)^k U^{-1}$ that is easy to observe because all products UU^{-1} cancel: $UU^{-1} = I$ in this expression.

One can also write $U^{-1}\exp\{A\}U = \exp\{U^{-1}AU\}$

One can hope to find new coordinates $y = \bar{U}^{-1}x$ such that $\exp(\bar{U}^{-1} A \bar{U})$ is easy to calculate that leads to a solution of the IVP in new coordinates:

$$\dot{y} = \left(\bar{U}^{-1} A \bar{U}\right) y \; ; \quad y(0) = y_0 .$$

The simplest case is when A has n linearly independent eigen vectors $\{u_k\}_{k=1}^{n}$. Choosing matrix $\bar{U} = (u_1 | u_2 | \dots | u_n)$ we get

$A\bar{U} = D \cdot \bar{U}$ with D - diagonal matrix with corresponding eigenvalues on the diagonal:

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}. \qquad \text{So} \quad \bar{U}^{-1} A \bar{U} = D .$$

$$\exp\{D\} = \begin{bmatrix} \exp(\lambda_1) & & 0 \\ & \exp(\lambda_2) & \\ 0 & & \ddots \; \exp(\lambda_n) \end{bmatrix} -$$

diagonal matrix with $\exp\{\lambda_k\}$ on the diagonal.

A useful observation is that $\exp\{B\}$ with B having block-diagonal structure

$$B = \begin{bmatrix} \boxed{B_1} & & & 0 \\ & \boxed{B_2} & & \\ & & \ddots & \\ 0 & & & \boxed{B_m} \end{bmatrix}$$

where B_k are square matrices and other elements of B are zero has the same structure:

$$\exp B = \begin{bmatrix} \boxed{\exp B_1} & & & \\ & \boxed{\exp B_2} & & \\ & & \ddots & \\ & & & \boxed{\exp B_m} \end{bmatrix}$$

It is valid because product of block-diagonal matrices of the same structure will have the same structure:

$$\begin{bmatrix} \boxed{A_1} & & & 0 \\ & \boxed{A_2} & & \\ & & \ddots & \\ 0 & & & \boxed{A_m} \end{bmatrix} \begin{bmatrix} \boxed{B_1} & & & 0 \\ & \boxed{B_2} & & \\ & & \ddots & \\ 0 & & & \boxed{B_m} \end{bmatrix} = \begin{bmatrix} \boxed{A_1B_1} & & & \\ & \boxed{A_2B_2} & & \\ & & \ddots & \\ & & & \boxed{A_mB_m} \end{bmatrix}$$

(blocks A_k, B_k, A_kB_k - have the same size for each k, but different for different k)

Jordan canonical form. Theorem 3.2

Let A be a complex $n \times n$ matrix. Then there exists a linear change of coordinates U such that $J = U^{-1}AU$ has a block-diagonal form

$$J = \begin{bmatrix} \boxed{J_1} & & & 0 \\ & \boxed{J_2} & & \\ 0 & & \ddots & \\ & & & \boxed{J_m} \end{bmatrix}$$ with each block J_k

of the form:

$$J_k = \alpha_k I + N = \begin{bmatrix} \alpha_k & 1 & & & 0 \\ & \alpha_k & 1 & & \\ & & \alpha_k & 1 & \\ & & & \ddots & 1 \\ 0 & & & & \alpha_k \end{bmatrix}$$

N has ones on the diagonal over the main diagonal and zeros elsewhere.

Numbers λ_k are eigenvalues of A. Columns u_j in the transformation matrix U are basis vectors for new coordinates: $x = Uy$. To observe what these vectors are we consider the matrix equation relating matrices A and J: $AU = UJ$, in particular columns i U, corresponding to a particular block J_k of size m.

$$A \underbrace{\begin{bmatrix} \cdots & u_l & u_{l+1} & \cdots & u_{l+m-1} & \cdots \\ & l & l+1 & & l+m-1 & \end{bmatrix}}_{\text{numbers of columns in } U} = \begin{bmatrix} \cdots & u_l & u_{l+1} & \cdots & u_{l+m-1} & \cdots \\ & l & l+1 & & & \end{bmatrix} \times$$

$$\times \begin{bmatrix} (1) & & & & & & (1) \\ & \lambda_k & 1 & 0 & 0 \cdots & 0 & \\ & 0 & \lambda_k & 1 & 0 \cdots & 0 & \\ (1) & 0 & 0 & \lambda_k & & 0 & (1) \\ & & & & \ddots & 1 & \\ & 0 & & & & \lambda_k & \\ & & & (1) & & & \\ & l & & & & l+m-1 & \end{bmatrix}$$

Rools for matrix multiplication imply that

$$Au_l = \lambda_k u_l$$

because each element of u_l vector in righthand side is multiplied by column with only λ_k non-zero

Similar use of matrix multiplication definition implies that

$$A u_{l+1} = \lambda_k u_{l+1} + u_l,$$

$$A u_{l+2} = \lambda_k u_{l+2} + u_{l+1},$$

$$\vdots$$

$$A u_{l+m-1} = \lambda_k u_{l+m-1} + u_{l+m-2},$$

$A u_l = \lambda_k u_l$

– u_l is an eigenvector corresponding to λ_k

or

$$(A - \lambda_k I) u_{l+1} = u_l \quad ;$$

$$(A - \lambda_k I) u_{l+2} = u_{l+1}$$

$$\vdots$$

$$(A - \lambda_k I) u_{l+m-1} = u_{l+m-2}$$

$$(A - \lambda_k I) u_l = 0$$

u_l - eigen vector corresponding λ_k.

The first basis vector corresponding to J_k is an eigenvector u_l, other basis vectors satisfy reccurent relations and can be computed from each other. These basis vectors are generalized eigenvectors to A: they satisfy relations:

$$(A - \lambda_k I)^{p+1} u_{l+p} = 0 \quad \text{(compare with}$$

$(A - \lambda_k I) u_l = 0$ for the eigenvector u_l.

Jordan Canonical Form

Theorem:(Jordan Canonical Form) Any constant $n \times n$ matrix A is similar to a matrix J in Jordan canonical form. That is, there exists an invertible matrix P such that the $n \times n$ matrix $J = P^{-1}AP$ is in the canonical form

$$J = \begin{bmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_s \end{bmatrix}.$$

where each Jordan block matrix J_k is an $n_k \times n_k$ matrix of the form

$$J_k = \begin{bmatrix} \lambda_k & 1 & 0 & \cdots & 0 \\ 0 & \lambda_k & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_k & 1 \\ 0 & 0 & \cdots & 0 & \lambda_k \end{bmatrix}, \quad (k = 1, 2, \ldots, s).$$

The sum $n_1 + n_2 + \cdots + n_s = n$. The numbers λ_k (k=1,2,...,s) are the eigenvalues of A. If $p \neq q$ and λ_p appears on the diagonal of J_p and λ_q appears on the diagonal of J_q, then λ_p need **not** be different from λ_q. In fact, if m_j denotes the geometric multiplicity of the eigenvalue λ_j of A, then λ_j will appear on the diagonal of exactly m_j blocks of J of the form J_j of differing sizes $(n_{j_1} \times n_{j_1}), \ldots, (n_{j_{m_j}} \times n_{j_{m_j}})$ and the sum $n_{j_1} + \ldots + n_{j_{m_j}} = r_j$, where r_j denotes the algebraic multiplicity of the eigenvalue λ_j.

The linearly independent columns of the matrix P such that $P^{-1}AP = J$ are chosen as follows:

Each column of P that corresponds to the first column of each Jordan block J_k, $k = 1, \ldots, s$ is an eigenvector of A corresponding to the eigenvalue λ_k. If we call these eigenvectors $p_{k,1}$, the remaining columns of P (if any) are made up of generalized eigenvectors of A arranged in order of increasing grade and related to each other by

$$(A - \lambda_k I)p_{k,g+1} = p_{k,g}, \quad g = 1, 2, .., n_k - 1,$$

where $p_{k,g}$ denotes a generalized eigenvector of grade g corresponding to λ_k.

Observations about block-diagonal matrices imply that

$$\exp\{J\} = \begin{bmatrix} \boxed{\exp\{J_1\}} & & & 0 \\ & \boxed{\exp J_2} & & \\ & & \ddots & \\ 0 & & & \boxed{\exp J_m} \end{bmatrix}$$

and one needs to calculate $\exp\{J_k\}$ of separate Jordan blocks to calculate $\exp\{J\}$. $J_k = \lambda_k I + N_k$

Matrix $N = \begin{bmatrix} 0 & 1 & & 0 \\ & 0 & 1 & \\ & & 0 & 1 \\ 0 & & & \ddots \end{bmatrix}$ is nilpotent:

$N_k^p = 0$ for N_k of size $p \times p$.

In general it is easy to observe that N

$$NA = \begin{bmatrix} 0 & \vdots & \vdots & & \vdots \\ 0 & a_1 & a_2 & \dots & a_{p-1} \\ 0 & \vdots & \vdots & & \vdots \end{bmatrix}$$

transforms arbitrary matrix A to one with columns shifted to the right and zeroes put in the first column. This observation implies that $N_k^p = 0$. I and N_k commute therefore $\exp\{J_k\} = e^{\lambda_k} \sum_{j=0}^{p-1} N_k^j \frac{1}{j!}$

where p is the size of the **block** N_k.

because higher order powers of N_k are all zero. Series for N_k are just finite sums.

We conclude that for J_k of size $p \times p$

$$\exp\{J_k\} = e^{\lambda_k}\begin{bmatrix} 1 & 1 & 1/2! & \cdots & \frac{1}{(p-1)!} \\ & 1 & 1 & \ddots & \vdots \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 1 \end{bmatrix}$$

One can also observe tha for an arbitrary analytical function $f(z)$ defined on all eigenvalues λ_k of A one can define $f(J)$ similarly by power series that reduce to

$$f\{J_k\} = \begin{bmatrix} f(\lambda_k) & f'(\lambda_k) & f''(\lambda_k)\frac{1}{2!} & \cdots & f^{(p-1)}(\lambda_k)\frac{1}{(p-1)!} \\ & \ddots & \ddots & \ddots & f^{(p-2)}(\lambda_k)\frac{1}{(p-2)!} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & \vdots \\ & & & & f(\lambda_k) \end{bmatrix}$$

Therefore by change of variables $Uy = x$ that transforms A to $J = U^{-1}AU$ we can compute

$$\exp\{A\} = U\exp\{J\}U^{-1}$$

With explicit expression for $\exp\{J\}$ in block-diagonal form. Number of Blocks in J is the same as the number of linearly independent eigenvectors to A.

Real Jordan canonical form.

The analysis before for canonical forms of matrices was for possibly complex matrices, complex eigenvalues and complex eigenvectors and generalized eigenvectors. If matrix A is real (the case most interesting to us) it still can have complex eigenvalues and eigenvectors. We like to get real solutions with real initial data in this case.

If λ_k is a complex eigenvalue to a real matrix A it comes together with its conjugate λ_k^*. Corresponding eigenvectors can be cosen also complex conjugate to each other u_k, u_k^*. We replace u_j and u_j^* by $\mathrm{Re}\, u_j$ and $\mathrm{Im}\, u_j$. In this basis the block $\begin{bmatrix} J & 0 \\ & J^* \end{bmatrix}$ in the Jordan normal form will be replaced by

$$\begin{bmatrix} R & I & & 0 \\ & R & I & \\ & & \ddots & I \\ 0 & & & R \end{bmatrix} \text{ with } R = \begin{bmatrix} \mathrm{Re}\,\lambda_k & \mathrm{Im}\,\lambda_k \\ -\mathrm{Im}\,\lambda_k & \mathrm{Re}\,\lambda_k \end{bmatrix};$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Matrices of the form ~~ka~~ R behave similarly to complex numbers $\mathrm{Re}\,\lambda_k + i\,\mathrm{Im}\,\lambda_k$. In particular $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ commute with each other.

The exponent $\exp\{J\}$ is computed similarly as above:

$$\exp\{J_k\} = \begin{pmatrix} \exp(R) & \exp(R) & \exp(R)\frac{1}{2!} & \dots & \exp(R)\frac{1}{(p-1)!} \\ & \exp(R) & \exp(R) & \ddots & \\ & & \ddots & \ddots & \\ & & & & \ddots \end{pmatrix}$$

With $\exp(R)$ computed after the Euler formula for complex numbers

$$\exp(\operatorname{Re} z + i \operatorname{Im} z) = \exp(\operatorname{Re} z)(\cos(\operatorname{Im} z) + i \sin(\operatorname{Im} z))$$

$$= \exp(\operatorname{Re} z)(\cos(\operatorname{Im} z) + i \operatorname{Im}(z))$$

$$\exp(R) = \exp(\operatorname{Re} \lambda_k)\begin{bmatrix} \cos(\operatorname{Im} \lambda_k) & \sin(\operatorname{Im} \lambda_k) \\ -\sin(\operatorname{Im} \lambda_k) & \cos(\operatorname{Im} \lambda_k) \end{bmatrix}$$

Classification and analysis of solutions to linear autonomous firs order systems of ODE.

Case of dimension 2 – plane systems. phase portraits in plane.

A – real 2×2 matrix. Possible Jordan canonical forms in this case are

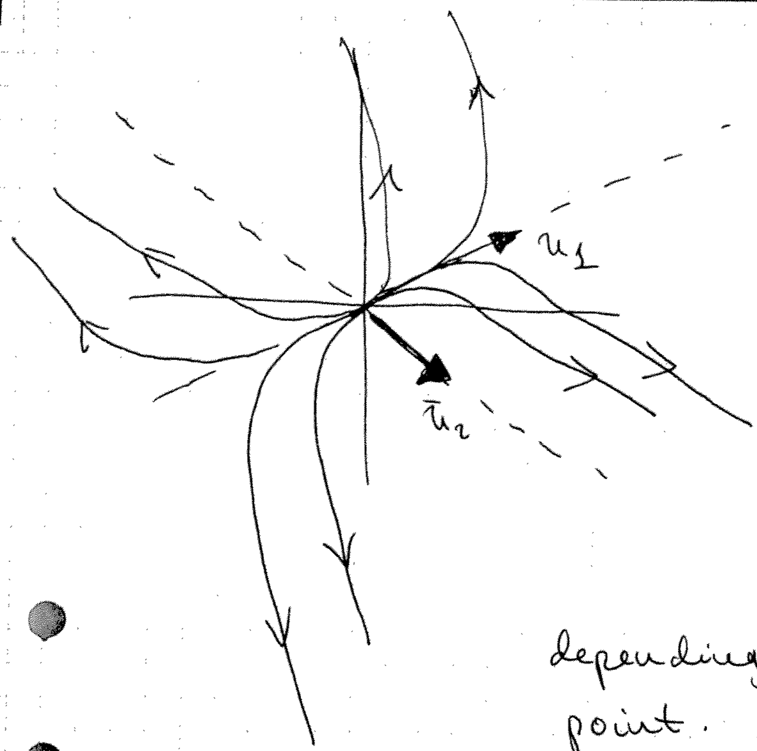
a) $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ $\lambda_1 > \lambda_2$; b) $\begin{bmatrix} \lambda_0, & 0 \\ 0 & \lambda_0 \end{bmatrix}$; $\lambda_1, \lambda_2, \lambda_0$ – real α, β,

c) $\begin{bmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{bmatrix}$ d) $\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$ $\beta < 0$.

Characteristic polynomial $P_A(\lambda)$ has the form
$$P_A(\lambda) = \lambda^2 - tr(A)\lambda + det(A) = 0$$
$tr A = A_{11} + A_{22}$; $det A = A_{11}A_{22} - A_{12}A_{21}$ for the 2×2 matrix A. Eigenvalues $\lambda_{1,2}$ are of the form
$$\lambda_1 = \frac{1}{2}\left(tr A + \sqrt{\Delta}\right), \quad \lambda_2 = \frac{1}{2}\left(tr A - \sqrt{\Delta}\right)$$
$$\left(\Delta = (tr(A))^2 - 4\,det(A)\right)$$

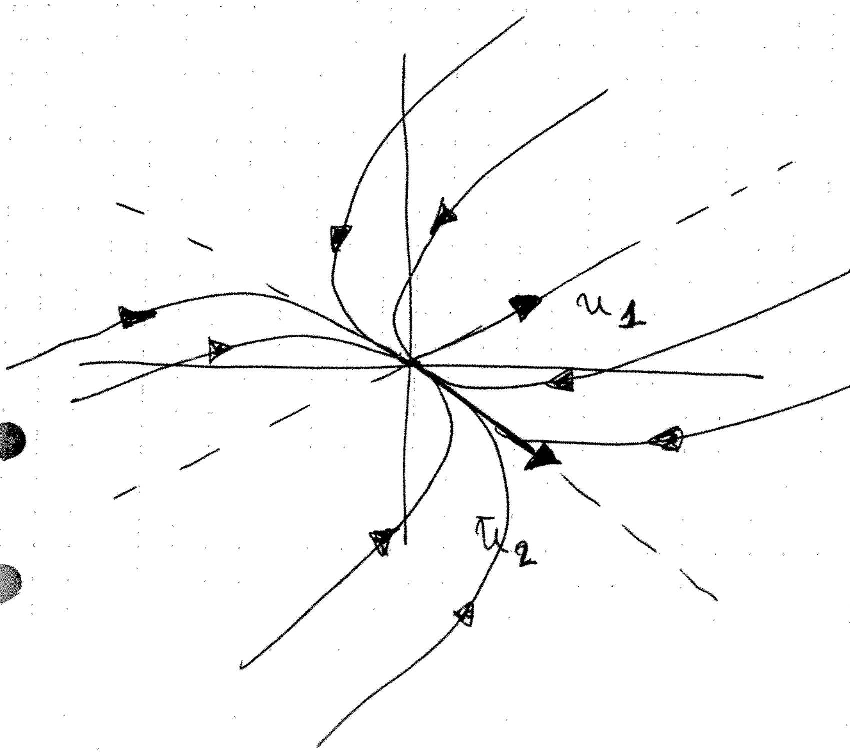
Problem 3.14 suggests to classify different types of solutions in the plane for different values of $tr(A)$ and $det(A)$. We will call a picture with a set of typical integral curves a phase portrait of the system.

If $det(A) < (tr(A))^2 \frac{1}{4}$ A has two different eigenvalues (real) α_1 and α_2 and two linearly independent eigenvectors u_1 and u_2.
$$U^{-1}AU = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}; \quad y(t) = U^{-1}x(t); \quad y_0 = U^{-1}x_0$$
$$\dot{y} = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} y; \quad y(0) = y_0$$
$$y(t) = \begin{pmatrix} y_{0,1}e^{\alpha_1 t} \\ y_{0,2}e^{\alpha_2 t} \end{pmatrix} = \begin{pmatrix} e^{\alpha_1 t} & 0 \\ 0 & e^{\alpha_2 t} \end{pmatrix} y_0 .$$
$$x(t) = U \begin{pmatrix} e^{\alpha_1 t} & 0 \\ 0 & e^{\alpha_2 t} \end{pmatrix} y_0, \quad \text{therefore}$$
$$x(t) = y_{01}e^{\alpha_1 t}u_1 + y_{02}e^{\alpha_2 t}u_2$$
Depending on signs of α_1, α_2 the phase portrait can be of three different types.



$0 < \alpha_1 < \alpha_2$

This configuration is called an unstable node or source, because both $x_1(t), x_2(t) \xrightarrow[t\to+\infty]{} \pm\infty$ depending on the starting point.



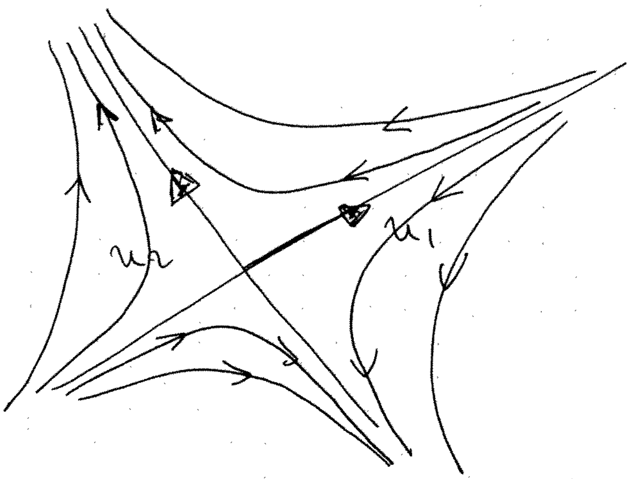
$\alpha_1 < \alpha_2 < 0$

This configuration is called a stable node or a sink, because $x_1(t) \to 0$, with $x_2(t) \to 0$ $t\to+\infty$

If $\alpha_1 < 0 < \alpha_2$ then the configuration of integral curves is different and is called a saddle. In this case

$$y_1(t) = y_{0,1} e^{\alpha_1 t} \underset{t \to +\infty}{\longrightarrow} 0$$

$$y_2(t) = y_{0,2} e^{\alpha_2 t} \underset{t \to +\infty}{\longrightarrow} \pm\infty$$



$x(t) = U y(t)$

u_1 and u_2 are columns in U and eigen vectors.

If A has two different complex (conjugate) eigenvalues with two complex conjugate eigen vectors $u_1 = u_2^*$, $\alpha_1 = \alpha_2^*$.

For $\alpha = \lambda + i\omega$ $\quad \exp(\alpha t) = e^{\lambda t}(\cos(\omega t) + i \sin \omega t)$

x_0 is real and $x_0^* = x_0$ and

$y_{0,1} u_1 + y_{0,2} u_2 = y_{0,1}^* u_2 + y_{0,2}^* u_1$,

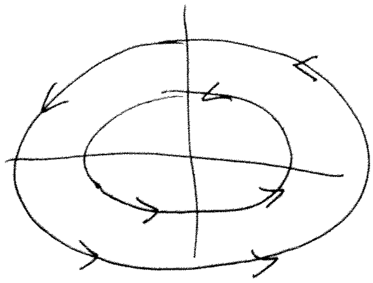
therefore $y_{0,1}^* = y_{0,2}$. Therefore solution formula $x(t) = y_{0,1} e^{\alpha_1 t} u_1 + y_{0,2} e^{\alpha_2 t} u_2$ implies that terms are complex conjugate and

$$x(t) = 2\,\mathrm{Re}\left(y_{0,1} e^{\alpha_1 t} u_1\right) =$$

$$= 2 e^{\lambda t} \cos(\omega t)\, \mathrm{Re}(y_{0,1} u_1) - 2 \sin(\omega t) e^{\lambda t}\, \mathrm{Im}(y_{0,1} u_1)$$

The phase portrait in this case consists of ellipses if $\lambda = \mathrm{Re}\, \alpha_{1,2} = 0$

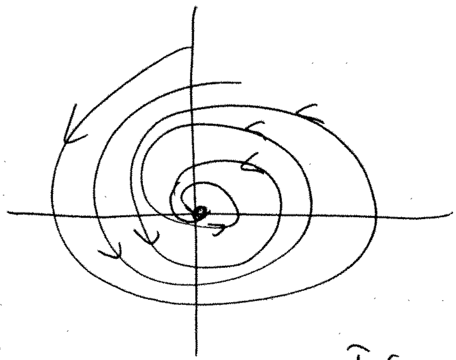
(configuration is called <u>center</u>)



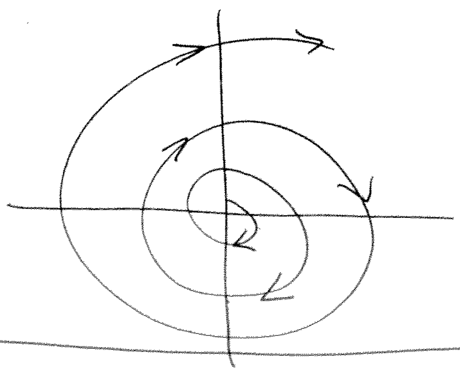
and spirals if $\lambda \neq 0$

If $\mathrm{Re}\, \alpha_{1,2} < 0$

integral curves are spirals tending to the origin: (<u>stable spirals</u>)



If $\mathrm{Re}\, \alpha_1, \alpha_2 > 0$, then integral curves are spirals tending out from the origin: <u>unstable spirals</u>.



If eigenvalues are equal an real (α) (they cannot be complex i 2-dimensions, one needs at least dimension 4 for that!)

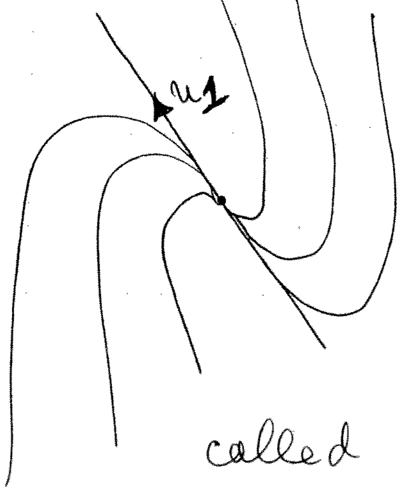
If A has only one linearly independent eigenvector u_1, the transformation U with u_1 - eigenvector and u_2 - satisfying equation

$$(A-\alpha I)u_2 = u_1 \quad \text{- its columns.}$$

$$U^{-1}AU = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$$ and solution

$$x(t) = (y_{0,1} + t y_{0,2}) e^{\alpha t} u_1 + y_{0,2} e^{\alpha t} u_2$$

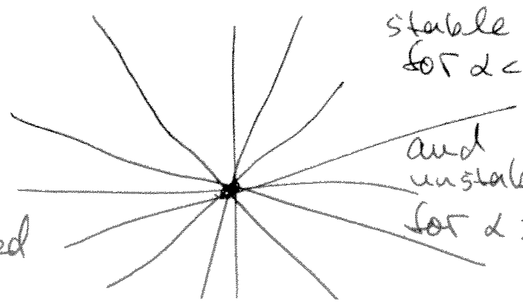
Phase portrait looks as



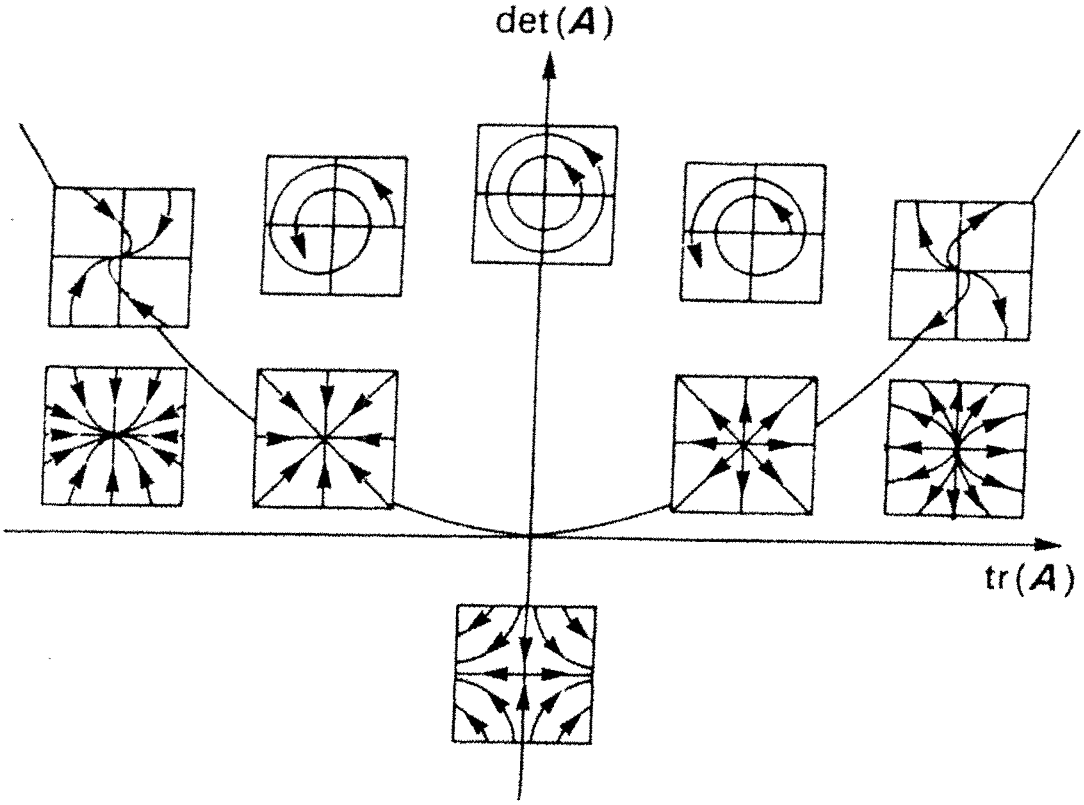
with integral curves tending to the origin for $\alpha < 0$ and out from the origin if $\alpha > 0$.

This configuration is called improper node (stable for $\alpha < 0$ and unstable for $\alpha > 0$)

If $\alpha_1 = \alpha_2$ - real and there are two eigenvectors u_1 and u_2 - linearly independent, the phase portrait is called star



stable for $\alpha < 0$ and unstable for $\alpha > 0$



Summary of phase portraits for the system x'=Ax depending on tr(A) and det(A).
The division line is det(A)= ¼ $(\mathrm{tr}(A))^2$

$$
\exp\{t J_k\} = e^{\alpha_k t}\begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{p-1}}{(p-1)!} \\ & 1 & t & \ddots & \vdots \\ & & 1 & \ddots & \frac{t^2}{2!} \\ & & & \ddots & t \\ & & & & 1 \end{bmatrix}
$$

Phase space decomposition and stability of solutions of linear autonomous systems in higher dimensions.

$y(t) = \exp\{tJ\} y_0$; J - Jordan block-diagonal canonical matrix $J = V^{-1}AV$.

$x(t) = V\exp\{t\cdot J\} y_0$;

$X(t) = Vy(t)$; $x_0 = Vy_0$.

Block-diagonal structure of J implies that the solution $x(t)$ corresponding to initial data x_0 from a subspace spanned by generalised eigenvectors in V, corresponding to a certain block J_k in J will belong to the same subspace as x_0. If λ_k - eigenvalue of J_k has $\mathrm{Re}\,\lambda_k < 0$, $x(t) \xrightarrow[t\to+\infty]{} 0$ because its components are polynomials of t times $e^{\lambda_k t}$ with $\lambda_k < 0$.

in case of real eigenvalue, or polinomials of t times $\cos(\mathrm{Im}\,\lambda_k t)\, e^{\mathrm{Re}\,\lambda_k t}$ or $\sin((\mathrm{Im}\,\lambda_k)t)\, e^{\mathrm{Re}\,\lambda_k t}$ if λ_k is complex

Subspaces of generalised eigenvectors, corresponding to the same eigenvalue λ_k (an several Jordan blocks, corresponding to linearly independent eigenvectors) are also invariant under action of $\exp\{At\}$.

These observations imply that following

Theorem 3.4 A solution of the linear system $\dot{x} = Ax$ with initial condition x_0 converges to zero as $t \to +\infty$ if and only if x_0 lies in a subspace spanned by generalized eigenvectors corresponding to eigenvalues with negative real part.
It will remain bounded if and only if x_0 lies in the subspace spanned by generalized eigenvectors corresponding to eigenvalues with negative and (eigenspaces of eigenvalues with) ~~zero~~ real part.

Corollary 3.5. (Definition first!)
A linear system $\dot{x} = Ax$ is stable if all solutions remain bounded as $t \to +\infty$ and asymptotically stable if all solutions $x(t) \underset{t \to +\infty}{\longrightarrow} 0$.

Linear system $\dot{x} = Ax$ is stable if and only if all eigenvalues α_k of A satisfy $Re(\alpha_k) \leq 0$ and all eigenvalues with $Re(\alpha_j) = 0$ algebraic and geometric multiplicities are equal. In the last case $\exists C$:

$$\|\exp At\| \leq C, \quad t \geq 0.$$

Corollary 3.6. The system $\dot{x} = Ax$ is asymptotically stable if and only if all eigenvalues α_k of A have $Re\,\alpha_k < 0$. There is a constant $C = C(\alpha)$

$$: \|\exp\{tA\}\| \leq C e^{-t\alpha}, \quad \alpha < \min\{-Re(\alpha_k)\}_{k=1}^{m}$$

Calculate e^{At} for $A =$

1) $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ 2) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ 3) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

4) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$;

Svar.

1) $\begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}$; 2) $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ 3) $\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$

4) $\begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$.