Suggested ODE problems from Chapter 1.

Problem 1.1. *Consider the case of a stone dropped from the height h. Denote by r the distance of the stone from the surface. The initial condition reads $r(0) = h$, $\dot{r}(0) = 0$. The equation of motion reads*

$$\ddot{r} = -\frac{\gamma M}{(R+r)^2} \qquad \textit{(exact model)}$$

respectively

$$\ddot{r} = -g \qquad \textit{(approximate model)},$$

where $g = \gamma M/R^2$ and R, M are the radius, mass of the earth, respectively.

(i) *Transform both equations into a first-order system.*

(ii) *Compute the solution to the approximate system corresponding to the given initial condition. Compute the time it takes for the stone to hit the surface ($r = 0$).*

(iii) *Assume that the exact equation also has a unique solution corresponding to the given initial condition. What can you say about the time it takes for the stone to hit the surface in comparison to the approximate model? Will it be longer or shorter? Estimate the difference between the solutions in the exact and in the approximate case. (Hints: You should not compute the solution to the exact equation! Look at the minimum, maximum of the force.)*

(iv) *Grab your physics book from high school and give numerical values for the case $h = 10m$.*

Problem 1.3. *Classify the following differential equations. Is the equation linear, autonomous? What is its order?*

(i) $y'(x) + y(x) = 0$.

(ii) $\frac{d^2}{dt^2}u(t) = t\,\sin(u(t))$.

(iii) $y(t)^2 + 2y(t) = 0$.

(iv) $\frac{\partial^2}{\partial x^2}u(x, y) + \frac{\partial^2}{\partial y^2}u(x, y) = 0$.

(v) $\dot{x} = -y$, $\dot{y} = x$.

Problem 1.4. *Which of the following differential equations for $y(x)$ are linear?*

(i) $y' = \sin(x)y + \cos(y)$.

(ii) $y' = \sin(y)x + \cos(x)$.

(iii) $y' = \sin(x)y + \cos(x)$.

Problem 1.5. *Find the most general form of a second-order linear equation.*

Problem 1.6. *Transform the following differential equations into first-order systems.*

(i) $\ddot{x} + t\,\sin(\dot{x}) = x$.

(ii) $\ddot{x} = -y$, $\ddot{y} = x$.

The last system is linear. Is the corresponding first-order system also linear? Is this always the case?

Problem 1.7. *Transform the following differential equations into autonomous first-order systems.*

(i) $\ddot{x} + t\,\sin(\dot{x}) = x$.

(ii) $\ddot{x} = -\cos(t)x$.

The last equation is linear. Is the corresponding autonomous system also linear?

Problem 1.8. *Let $x^{(k)} = f(x, x^{(1)}, \ldots, x^{(k-1)})$ be an autonomous equation (or system). Show that if $\phi(t)$ is a solution, then so is $\phi(t - t_0)$.*

Problem 1.9. *Solve the following differential equations:*

(i) $\dot{x} = x^3$.

(ii) $\dot{x} = x(1-x)$.

(iii) $\dot{x} = x(1-x) - c$.

Problem 1.11 (Separable equations). *Show that the equation ($f, g \in C^1$)*

$$\dot{x} = f(x)g(t), \qquad x(t_0) = x_0,$$

locally has a unique solution if $f(x_0) \neq 0$. Give an implicit formula for the solution.

Problem 1.14. *Charging a capacitor is described by the differential equation*

$$R\dot{Q}(t) + \frac{1}{C}Q(t) = V_0,$$

where $Q(t)$ is the charge at the capacitor, C is its capacitance, V_0 is the voltage of the battery, and R is the resistance of the wire.

Compute $Q(t)$ assuming the capacitor is uncharged at $t = 0$. What charge do you get as $t \to \infty$?

Problem 1.15 (Growth of bacteria). *A certain species of bacteria grows according to*

$$\dot{N}(t) = \kappa N(t), \qquad N(0) = N_0,$$

where $N(t)$ is the amount of bacteria at time t, $\kappa > 0$ is the growth rate, and N_0 is the initial amount. If there is only space for $N_{\max}$ bacteria, this has to be modified according to

$$\dot{N}(t) = \kappa(1 - \frac{N(t)}{N_{\max}})N(t), \qquad N(0) = N_0.$$

Solve both equations, assuming $0 < N_0 < N_{\max}$ and discuss the solutions. What is the behavior of $N(t)$ as $t \to \infty$?

Problem 1.16 (Optimal harvest). *Take the same setting as in the previous problem. Now suppose that you harvest bacteria at a certain rate $H > 0$. Then the situation is modeled by*

$$\dot{N}(t) = \kappa(1 - \frac{N(t)}{N_{\max}})N(t) - H, \qquad N(0) = N_0.$$

Rescale by

$$x(\tau) = \frac{N(t)}{N_{\max}}, \qquad \tau = \kappa t$$

and show that the equation transforms into

$$\dot{x}(\tau) = (1 - x(\tau))x(\tau) - h, \qquad h = \frac{H}{\kappa N_{\max}}.$$

Visualize the region where $f(x,h) = (1-x)x - h$, $(x,h) \in U = (0,1) \times (0,\infty)$, *is positive respectively negative. For given* $(x_0, h) \in U$, *what is the behavior of the solution as* $t \to \infty$*? How is it connected to the regions plotted above? What is the maximal harvest rate you would suggest?*

Lemma 1.1. *Consider the first-order autonomous initial value problem* (1.61)*, where* $f \in C(\mathbb{R})$ *is such that solutions are unique.*

(i) *If* $f(x_0) = 0$*, then* $x(t) = x_0$ *for all* t*.*

(ii) *If* $f(x_0) \neq 0$*, then* $x(t)$ *converges to the first zero left (*$f(x_0) < 0$*) respectively right (*$f(x_0) > 0$*) of* x_0*. If there is no such zero the solution converges to* $-\infty$*, respectively* ∞*.*

Problem 1.27. *Let* x *be a solution of* (1.61) *which satisfies* $\lim_{t\to\infty} x(t) = x_1$*. Show that* $\lim_{t\to\infty} \dot{x}(t) = 0$ *and* $f(x_1) = 0$*. (Hint: If you prove* $\lim_{t\to\infty} \dot{x}(t) = 0$ *without using* (1.61) *your proof is wrong! Can you give a counter example?)*

Problem 1.28. *Prove Lemma 1.1. (Hint: This can be done either by using the analysis from Section 1.3 or by using the previous problem.)*

Problem 1.32. *Generalize Theorem 1.3 to the interval* (T, t_0)*, where* $T < t_0$*.*

To complete our analysis suppose $h < h_c$ and denote by $x_1 < x_2$ the two fixed points of $P(x)$. Define the iterates of $P(x)$ by $P^0(x) = x$ and $P^n(x) = P(P^{n-1}(x))$. We claim

$$\lim_{n\to\infty} P^n(x) = \begin{cases} x_2, & x > x_1, \\ x_1, & x = x_1, \\ -\infty, & x < x_1. \end{cases} \tag{1.84}$$

Problem 1.33. *Suppose $P(x)$ is a continuous, monotone, and concave function with two fixed points $x_1 < x_2$. Show the remaining cases in* (1.84).

Problem 1.34. *Find $\lim_{n\to\infty} P^n(x)$ in the case $h = h_c$ and $h > h_c$.*

Problem 1.35. *Suppose $f \in C^2(\mathbb{R})$ and $g \in C(\mathbb{R})$ is a nonnegative periodic function $g(t+1) = g(t)$. Show that the above discussion still holds for the equation*

$$\dot{x} = f(x) + h \cdot g(t)$$

if $f''(x) < 0$ and $g(t) \geq 0$.

Problem 1.36. *Suppose $a \in \mathbb{R}$ and $g \in C(\mathbb{R})$ is a nonnegative periodic function $g(t+1) = g(t)$. Find conditions on a, g such that the linear inhomogeneous equation*

$$\dot{x} = ax + g(t)$$

has a periodic solution. When is this solution unique? (Hint: (1.40)*.)*

Problem 1.20. *Pick some differential equations from the previous problems and solve them using your favorite computer algebra system. Plot the solutions.*

Problem 1.30. *Discuss the equation $\dot{x} = x^2 - \frac{t^2}{1+t^2}$.*

- *Make a numerical analysis.*
- *Show that there is a unique solution which asymptotically approaches the line $x = 1$.*
- *Show that all solutions below this solution approach the line $x = -1$.*
- *Show that all solutions above go to ∞ in finite time.*