Problems on linear autonomous ODE problems.

9. Sketch phase portraits for the linear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

when A is given by:

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$; (b) $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$; (c) $\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$;

(d) $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$; (e) $\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$; (f) $\begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$;

(g) $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$; (h) $\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$; (i) $\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$;

(j) $\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$.

10. Indicate the effect of the linear transformation

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

on each of the systems in Exercise 2.9 by sketching each phase portrait in the $y_1 y_2$-plane.

11. Find the 2×2 matrix A such that the system

$$\dot{\mathbf{x}} = A\mathbf{x}$$

has a solution curve

$$\mathbf{x}(t) = \begin{bmatrix} e^{-t}(\cos t + 2\sin t) \\ e^{-t}\cos t \end{bmatrix}.$$

13. Locate each of the following linear systems in the Tr–Det plane and hence state their phase portrait type:

(a) $\dot{x}_1 = 2x_1 + x_2, \quad \dot{x}_2 = x_1 + 2x_2;$

(b) $\dot{x}_1 = 2x_1 + x_2, \quad \dot{x}_2 = x_1 - 3x_2;$

(c) $\dot{x}_1 = x_1 - 4x_2, \quad \dot{x}_2 = 2x_1 - x_2;$

(d) $\dot{x}_1 = 2x_2, \quad \dot{x}_2 = -3x_1 - x_2,$

(e) $\dot{x}_1 = -x_1 + 8x_2, \quad \dot{x}_2 = -2x_1 + 7x_2.$

21. Calculate e^{At} for the following matrices A:

(a) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$, (c) $\begin{bmatrix} 2 & 4 \\ 3 & 3 \end{bmatrix}$,

(d) $\begin{bmatrix} -2 & 2 \\ -4 & -2 \end{bmatrix}$, (e) $\begin{bmatrix} -4 & 1 \\ -1 & -2 \end{bmatrix}$,

22. Calculate e^{At} when A is equal to:

(a) $\begin{bmatrix} 2 & -7 \\ 3 & -8 \end{bmatrix}$; (b) $\begin{bmatrix} 2 & 4 \\ -2 & 6 \end{bmatrix}$.

34. Find the form of the solution curves of the systems $\dot{x} = Ax$, $x \in \mathbb{R}^4$, where A equals:

(a) $\begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$; (b) $\begin{bmatrix} \alpha & -\beta & 0 & 0 \\ \beta & \alpha & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$; (c) $\begin{bmatrix} \alpha & -\beta & 0 & 0 \\ \beta & \alpha & 0 & 0 \\ 0 & 0 & \gamma & -\delta \\ 0 & 0 & \delta & \gamma \end{bmatrix}$.

Svar:

11. $\dot{x}_1 = -\frac{3}{2}x_1 + \frac{5}{2}x_2,\ \dot{x}_2 = -\frac{1}{2}x_1 - \frac{1}{2}x_2.$

12. $x_1(t) = (2e^{-2t} - e^{-3t})x_1(0) + (e^{-3t} - e^{-2t})x_2(0);$
$x_2(t) = (2e^{-2t} - 2e^{-3t})x_1(0) + (-e^{-2t} + 2e^{-3t})x_2(0).$
Put $y_1 = 4e^{-2t} = x_1 + 3x_2$. Compare coefficients of e^{-2t} and e^{-3t} and obtain simultaneous equations for $x_1(0)$ and $x_2(0)$ $[x_1(0) = x_2(0) = 1]$.

13. (a) unstable node; (b) saddle; (c) centre; (d) stable focus; (e) unstable improper node.

21. (a) $\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix}$; (b) $\begin{bmatrix} e^{t} & 2e^{t}(e^{t}-1) \\ 0 & e^{2t} \end{bmatrix}$;

(c) $\frac{e^{6t}}{7}\begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix} + \frac{e^{-t}}{7}\begin{bmatrix} 4 & -4 \\ -3 & 3 \end{bmatrix}$;

(d) $\frac{e^{-2t}}{\sqrt{2}}\begin{bmatrix} \sqrt{2}\cos\beta t & \sin\beta t \\ -2\sin\beta t & \sqrt{2}\cos\beta t \end{bmatrix}$, $\beta = 2\sqrt{2}$;

(e) $e^{-3t}\begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix}$.

22. (a) $\frac{e^{-t}}{4}\begin{bmatrix} 7 & -7 \\ 3 & -3 \end{bmatrix} + \frac{e^{-5t}}{4}\begin{bmatrix} -3 & 7 \\ -3 & 7 \end{bmatrix}$;

(b) $e^{4t}\begin{bmatrix} \cos 2t - \sin 2t & 2\sin 2t \\ -\sin 2t & \cos 2t + \sin 2t \end{bmatrix}$.

34. $e^{At} = :$

(a) $$e^{\lambda t}\begin{bmatrix} 1 & t & \frac{t^2}{2} & \frac{t^3}{6} \\ 0 & 1 & t & \frac{t^2}{2} \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) $$\left[\begin{array}{c|c} e^{\alpha t}\begin{bmatrix} \cos\beta t & -\sin\beta t \\ \sin\beta t & \cos\beta t \end{bmatrix} & \boldsymbol{0} \\ \hline \boldsymbol{0} & e^{\lambda t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \end{array}\right].$$

(c) $$\left[\begin{array}{c|c} e^{\alpha t}\begin{bmatrix} \cos\beta t & -\sin\beta t \\ \sin\beta t & \cos\beta t \end{bmatrix} & \boldsymbol{0} \\ \hline \boldsymbol{0} & e^{\gamma t}\begin{bmatrix} \cos\delta t & -\sin\delta t \\ \sin\delta t & \cos\delta t \end{bmatrix} \end{array}\right]$$