4. Find the linearizations of the following systems, at the fixed points indicated, by:

(a) introducing local coordinates at the fixed point;

(b) using Taylor's theorem.

(i) $\dot{x}_1 = x_1 + x_1 x_2^3/(1+x_1^2)^2, \quad \dot{x}_2 = 2x_1 - 3x_2, \quad (0, 0)$;

(ii) $\dot{x}_1 = x_1^2 + \sin x_2 - 1, \quad \dot{x}_2 = \sinh(x_1 - 1), \quad (1, 0)$;

(iii) $\dot{x}_1 = x_1^2 - e^{x_2}, \quad \dot{x}_2 = x_2(1 + x_2), \quad (e^{-1/2}, -1)$.

State the preferred method (if one exists) for each system.

5. Use the linearization theorem to classify, where possible, the fixed points of the systems:

(a) $\dot{x}_1 = x_2^2 - 3x_1 + 2, \quad \dot{x}_2 = x_1^2 - x_2^2$;

(b) $\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + x_1^3$;

(c) $\dot{x}_1 = \sin(x_1 + x_2), \quad \dot{x}_2 = x_2$;

(d) $\dot{x}_1 = x_1 - x_2 - e^{x_1}, \quad \dot{x}_2 = x_1 - x_2 - 1$;

(e) $\dot{x}_1 = -x_2 + x_1 + x_1 x_2, \quad \dot{x}_2 = x_1 - x_2 - x_2^2$;

(f) $\dot{x}_1 = x_2, \quad \dot{x}_2 = -(1 + x_1^2 + x_1^4)x_2 - x_1$;

(g) $\dot{x}_1 = -3x_2 + x_1 x_2 - 4, \quad \dot{x}_2 = x_2^2 - x_1^2$.

Answers:

4. (i) $\dot{y}_1 = y_2, \dot{y}_2 = 2y_1 - 3y_2$;

(ii) $\dot{y}_1 = 2y_1 + y_2, \dot{y}_2 = y_1$;

(iii) $\dot{y}_1 = 2e^{-\frac{1}{2}} y_1 - e^{-1} y_2, \dot{y}_2 = -y_2$.

5. (a) $(1, -1)$, saddle; $(1, 1)$, stable node; $(2, -2)$, unstable focus; $(2, 2)$, saddle.

(b) $(\pm 1, 0)$, saddle; $(0, 0)$, centre;

(c) $(m\pi, 0)$, unstable improper node (m even), saddle (m odd);

(d) $(0, -1)$, stable focus;

(e) $(0, 0)$, non-simple; $(2, -2)$, saddle;

(f) $(0, 0)$, stable focus;

(g) $(-1, -1)$, stable focus; $(4, 4)$, unstable focus.