Stability by linearization for autonomous and non-autonomous systems. Perturbations of linear systems.

Two kinds of questions appear in application an in theory.

1) Compare stability properties of a linear system

$$\dot{x} = A(t)x$$

and its perturbation by a reasonably bounded nonlinear function $g(t,x)$:

$$\dot{x} = A(t)x + g(t,x)$$

2) Investigate stability of a fixed point x_0 of a nonlinear autonomous system

$$\dot{x} = f(x)$$

by comparing it with its linearization

$$\dot{y} = Ay$$

where A is a Jacobi matrix of the function f around the fixed point x_0.

We consider first the first type of problem.

Theorem 3.26. Consider system

$$\dot{x} = A(t)\cdot x + g(t,x),$$

$A(t)$ – $n\times n$ matrix $A\in C(\mathbb{R},\mathbb{R}^{n\times n})$ and $g(t,x)\in C([0,+\infty),\mathbb{R}^n)$, (Lipschitz in x) and principal matrix for linear system $\dot{x} = A(t)x$ satisfies the estimate

$$\|\Pi_A(t,s)\| \le C e^{-\alpha(t-s)} \qquad t\geq s\geq 0$$

for some constants $C,\alpha>0$. Suppose that

$$|g(t,x)| \le b_0|x| \quad \text{for } |x|<\delta,\ t\geq 0$$

for some $0<\delta<\infty$.

Then if $b_0 C<\alpha$, then the solution $x(t)$ starting at $x(0)=x_0$ satisfies

$$|x(t)| \le D e^{-(\alpha-b_0C)t}|x_0| \quad \text{for } |x_0|<\frac{\delta}{C},$$

$t\geq 0$ for some constant D.

Proof. We use a variant of Duhamel formula 3.97 for inhomogeneous linear systems ODE:

$$\dot{x}(t) = A(t)x(t) + g(t), \quad x(t_0)=x_0$$

$$x(t) = \Pi(t,t_0)x_0 + \int_{t_0}^{t}\Pi(t,s)g(s)\,ds$$

We interpret the nonlinear function $g(t,x)$ (3) as a given right hand side for a while and get an expression:

$$x(t) = \Pi_A(t,t_0)\,x_0 + \int_{t_0}^{t} \Pi_A(t,s)\,b(s,x(s))\,ds,$$

where $\Pi_A(t,t_0)$ is a principal matrix solution to the linear system (homogeneous):

$$\dot{x}(t) = A(t)\,x(t).$$

Assumptions on $A(t)$ and $b(t,x)$ imply the estimate for $|x(t)|$:

$$|x(t)| \leq C\, e^{-\alpha(t-s)} |x(s)| + \int_{s}^{t} C e^{-\alpha(t-r)}\, b_0 |x(r)|\,dr$$

$$t \geq s \geq t_0$$

With the notation $y(t) = |x(t)|\, e^{\alpha(t-s)}$ we get

$$y(t) \leq C\,|x(s)| + \int_{s}^{t} (C b_0)\, y(r)\,dr$$

Gronwall's inequality follows for $y(t)$:

$$|y(t)| \leq C\,|x(s)| \cdot e^{C b_0 (t-s)},$$

and the estimate for $|x(t)|$ follows:

$$|x(t)| \leq C\,|x_0|\, e^{-(\alpha - b_0 C)t},$$

for $s=0$; $x(0) = x_0$, $|x_0| < \dfrac{\delta}{C}$, $t \geq 0$

(The last assumptions implies that $|x(t)| \leq \delta$ that is necessary for the estimate for $|b(s,x)| \leq |b_0|\,|x|$)

Stability of fixed points of autonomous systems by linearization.

We consider the case $f(0)=0$, $t_0=0$. By the change of variables one can reduce the general case to this one.

<u>Corollary 3.27.</u> (Appears again as Theorem 6.10 later)

Consider I.V.P. $\dot{x}=f(x)$, $x(0)=x_0$, $f(0)=0$ (the origin is a fixed point)

Suppose $f\in C^1$ in a ball around the origin. Suppose that the Jacobian matrix A at 0 has all eigenvalues with negative real part: $\operatorname{Re}\lambda_j<-\beta$, $\beta>0$.

Then $\exists\,\delta>0$, $C>0$, $\alpha>0$ such that solutions to $\dot{x}=f(x)$, $x(0)=x_0$ satisfy the estimate:

$$|x(t)|\le C e^{-\alpha t}|x_0| \quad \text{for } |x_0|<\delta$$

Later we will say that the origin is asymptotically stable (even exponentially stable) fixed point.

Proof to corollary 3.27.

Express $f(x) = A \cdot x + g(x)$ where A is the Jacobi matrix of f in the origin. According to Taylor expansion of $f \in C^1$ $g(x) = o(|x|)$ when $|x| \to 0$. It means that $g(x) = |x| \cdot \xi(x)$ with $\lim_{x \to 0} \xi(x) = 0$. Therefore we can

for any $\varepsilon > 0$ find $\delta_\varepsilon > 0$ such that for any $|x| < \delta_\varepsilon \Rightarrow |\xi(x)| < \varepsilon$

If all eigenvalues $\lambda_j < -\beta$ then theory of linear systems with constant coefficients (Corollary 3.6)

$$\|\exp(tA)\| \leq C e^{-t\beta}$$ for some constant $C > 0$. We can now choose $\varepsilon > 0$ such that $\varepsilon \cdot C < \beta$ and corresponding δ_ε such that for $|x| < \delta_\varepsilon$ $|\xi(x)| < \varepsilon$.

Now the estimate
$$|x(t)| \leq C |x_0| e^{-(\beta - \varepsilon C)t}$$

follows by Theorem 3.26 and we can choose $\alpha = \beta - \varepsilon C$ to finish the proof. $\boxed{}$