Examples from ecology.

Volterra-Lotka predator-pray model. Predator-pray with limited growth.

x - number of prays

y - number of predators.

$$\begin{cases} \dot{x} = (A - By)\,x \\ \dot{y} = (Cx - D)\,y \end{cases} \qquad A, B, C, D > 0$$

Scaling x, y, t leads to one-parameter system:

$$\begin{cases} \dot{x} = (1-y)\,x \\ \dot{y} = \alpha(x-1)\,y \end{cases} \qquad \alpha > 0$$

Two fixed points $\to (0,0)$; $(1,1)$

Lines $x=0$ and $y=0$ are invariant sets:

$$\Phi(t, (0,y)) = (0, y e^{-\alpha t})$$

$$\Phi(t, (x,0)) = (x e^{t}, 0)$$

One cannot solve the system, but we can get an implicit expression for orbits.

$$\frac{dy}{dx} = \frac{dy}{dt}\left(\frac{dx}{dt}\right)^{-1} = \alpha\,\frac{(x-1)\,y}{(1-y)\,x}$$

The set $x \geq 0$, $y \geq 0$ is a positive invariant set for the system.

This is an equation with separable variables.

$$\frac{(1-y)}{y}\,dy = \alpha\left(\frac{x-1}{x}\right)dx$$

Integration implies

$$f(x) + \alpha f(y) = \text{const} = E \qquad (*)$$

där $f(x) = \int \frac{1-x}{x}\,dx = (x-1) - \ln(x)$

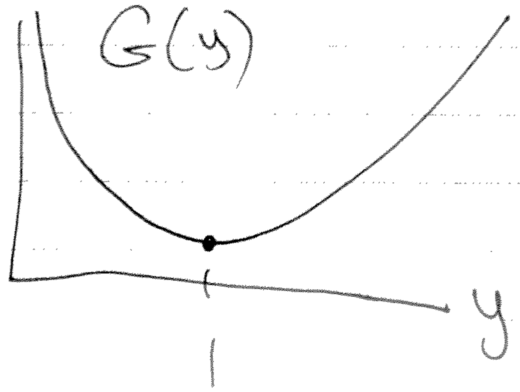
Taking exp $(*)$ we get

$$\frac{1}{y}e^{y-1} = e^{-\alpha(x-1)}\,x^{\alpha}\cdot e^{E}$$

Let $G(y) = \frac{1}{y}e^{y-1}$; $G'(y) = e^{y-1}\left(\frac{y-1}{y^2}\right)$

=> G has minimum in $y = 0$

$\underset{y \to 0+}{G(y) \to +\infty}$; $\underset{y \to +\infty}{G(y) \to +\infty}$

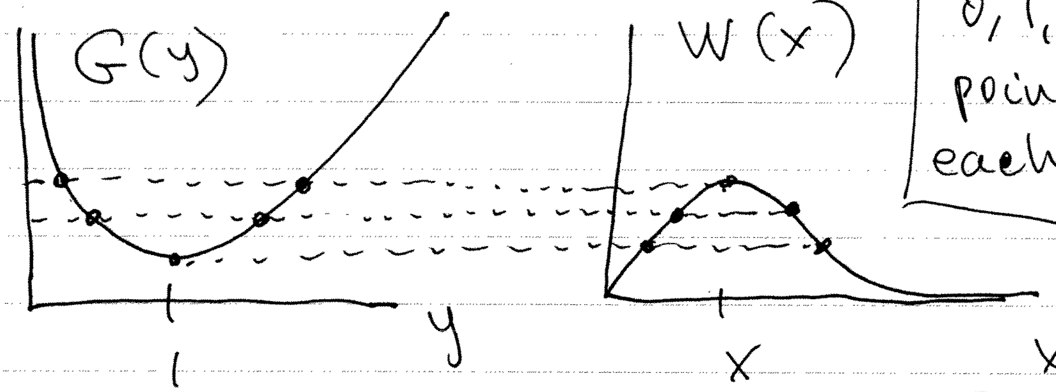


Let $W(x) = e^{-\alpha(x-1)} x^{\alpha} \cdot e^{E}$

$W(0)=0$; $W(x) \xrightarrow[x \to +\infty]{} 0$;

$$W'(x) = e^{-\alpha(x-1)}(-\alpha)x^{\alpha}e^{E} + \alpha x^{\alpha-1}e^{-\alpha(x-1)}e^{E} = e^{E}\cdot\alpha x^{\alpha-1}e^{-\alpha(x-1)}(1-x)$$

$W'(1)=0$ and $x=1$ is a local maximum for W.



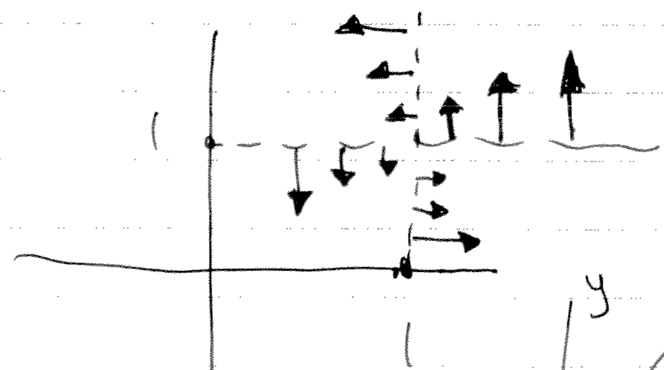
0, 1, or 2 points for each $G=W$

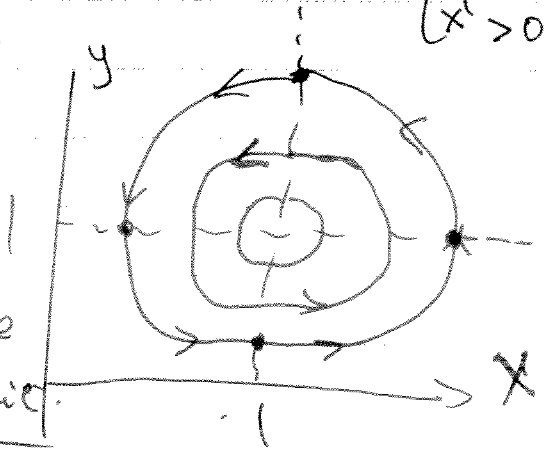
Orbits of the system are zero level sets of the function $G(y)-W(x)=0$

Isoclines $x=1$, $y=1$ give directions of trajectories:

$$y=1 \Rightarrow \begin{cases} y'<0, & x<1 \\ y'>0, & x>1 \end{cases}$$

$$x=1 \Rightarrow \begin{cases} x'<0, & y>1 \\ x'>0, & y<1 \end{cases}$$





Using graphs for W and G and isoclines we observe that orbits are periodic.

Model with limited growth.

$$\dot{x} = (1 - y - \lambda x)\, x$$
$$\dot{y} = \alpha\,(x - 1 - \mu y)\, y \qquad , \alpha, \lambda, \mu > 0$$

Species compete within themself $\lambda, \mu > 0$.

There are 4 fixed points: $(0,0)$, $(\lambda^{-1}, 0)$, $(0, -\mu^{-1})$, $\left(\frac{1+\mu}{1+\mu\lambda}, \frac{1-\lambda}{1+\mu\lambda}\right)$

The third point is outside the physically relevant invariant set $x \geq 0$, $y \geq 0$. The fourth fixed point is relevant only if $0 < \lambda < 1$. We consider only this case and are interested in the qualitative properties of phase portrait.

In the case of Lotka-Volterra equation the fixed point was neutrally (not asymptotically) stable and was surrounded by periodic orbits filling all the domain $x > 0$, $y > 0$. We are going to show that in the present case an arbitrarily small perturbation with $\lambda > 0$, $\mu > 0$ - small numbers, the pase portrait will be different: the fixed point $\left(\frac{1+\mu}{1+\mu\lambda}, \frac{1-\lambda}{1+\mu\lambda}\right)$ is asymptotically stable and attracts all trajectories starting in $x>0$, $y>0$.

Consider a test function

$$L(x,y) = \gamma_1 f\left(\frac{y}{y_0}\right) + \alpha \gamma_2 f\left(\frac{x}{x_0}\right)$$

Let $\bar{x} = x - x_0$, $y = y - y_0$

$$x_0 = \left(\frac{1+\mu}{1+\mu\lambda}\right); \quad y_0 = \left(\frac{1-\lambda}{1+\mu\lambda}\right)$$

The equations are transformed into

$$\dot{x} = (-\bar{y} - \lambda\bar{x})x; \quad \dot{y} = \alpha(\bar{x} - \mu\bar{y})y$$

$$(L)' = \frac{\partial L}{\partial x}\dot{x} + \frac{\partial L}{\partial y}\cdot\dot{y} =$$

$$= -\alpha\left(\frac{\lambda\gamma_2}{x_0}(\bar{x})^2 + \frac{\mu\gamma_1}{y_0}(\bar{y})^2 + \left(\frac{\gamma_2}{x_0} - \frac{\gamma_1}{y_0}\right)\bar{x}\bar{y}\right)$$

Choosing $\gamma_1 = y_0$; $\gamma_2 = x_0$ we get $\left(L'(x(t), y(t))\right) < 0$ $(x,y) \neq (x_0, y_0)$

Therefore L is a strict Liapunov's function, (x_0, y_0) is asymptotically stable and all trajectories are attracted to it.

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$$(1-y-\lambda x) = (1-(y-y_0) - y_0 - \lambda(x-x_0) - \lambda x_0 =$$

$$= -\bar{y} - \lambda\bar{x} + 1 - \frac{1-\lambda}{1+\mu\lambda} - \left(\frac{1+\mu}{1+\mu\lambda}\right)\lambda =$$

$$= -\bar{y} - \lambda\bar{x} + \frac{\cancel{1} + \cancel{\mu\lambda} - \cancel{1} + \cancel{\lambda} - \cancel{\lambda} - \cancel{\mu\lambda}}{1+\mu\lambda} =$$

$$= -y - \lambda\bar{x}$$

$$x - 1 - \mu y = (x-x_0) + x_0 - 1 - \mu(y-y_0) - \mu y_0 =$$

$$= \bar{x} - \mu\bar{y} + \frac{\cancel{-1} - \cancel{\mu\lambda} + \cancel{1} + \cancel{\mu} - \cancel{\mu} + \cancel{\mu\lambda}}{1+\mu\lambda} =$$

$$\dot{L} = \frac{\partial L}{\partial x}\cdot\dot{x} + \frac{\partial L}{\partial y}\cdot\dot{y} =$$

$$= \alpha\gamma_2\left(\frac{\bar{x}}{\cancel{x}x_0}\right)\left(\cancel{x}\left(-\bar{y} - \lambda\bar{x}\right)\right) +$$

$$+\alpha\gamma_1\left(\frac{\bar{y}}{\cancel{y}y_0}\right)\left(\bar{x} - \mu\bar{y}\right)\cancel{y} =$$

$$= -\frac{\alpha\lambda}{x_0}\gamma_2(\bar{x})^2 - \frac{\alpha\gamma_1\mu}{y_0}(\bar{y})^2$$

$$-\frac{\alpha\gamma_2}{x_0}(\bar{x}\bar{y}) + \frac{\alpha\gamma_1}{y_0}\bar{x}\bar{y} \; *$$