Well posedness of ODE. Dependence on initial data and on right hand side. Extensibility.

An ODE is called "Well posed" if small changes in "initial data and small perturbations in the right hand side in the equation lead to small changes in the solution.

The following theorem gives a quantitative meaning to this notion.

Theorem 2.8 Let $f, g \in C(U, \mathbb{R}^n)$, $U \subset \mathbb{R}^{n+1}$ - open, and f - locally Lipschitz

Consider two equations

$\dot{x} = f(t,x)$, $\dot{y} = g(t,y)$ with initial data $x(t_0) = x_0$, $y(t_0) = y_0$.

Let $L = \mathrm{Lip}(f)$; $M = \sup_{(t,x)\in V} |f - g|$

$V \subset U$, that graphs of $x(t)$ and $y(t)$ belong to V.

Then

$$|x(t) - y(t)| \le |x_0 - y_0| e^{L(t-t_0)} + \frac{M}{L}\left(e^{L(t-t_0)} - 1\right)$$

Proof in the book uses a simpler variant of Gronwall's enequality (not proved in the book)

We give both proofs together.

Proof. We use integral form of equations

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\,ds$$

$$y(t) = y_0 + \int_{t_0}^{t} g(s, y(s))\,ds$$

The implies

$$|x(t) - y(t)| \le |x_0 - y_0| + \int_0^t |f(s, x(s)) - g(s, y(s))|\,ds$$

$$|f(s, x(s)) - g(s, y(s))| \le |f(s, x(s)) - f(s, y(s)) +$$

$$f(s, y(s)) - g(s, y(s))| \le$$

$$\le L\,|x(s) - y(s)| + M$$

We denote $r(t) = |x(t) - y(t)|$ and get

$$\boxed{r(t) \le r(t_0) + \int_{t_0}^{t} L\,r(s)\,ds + Mt}$$

We consider first a simpler case when $f = g$, $M = 0$.

Lemma. Let $r(t) \le c + \int_0^t L\,r(s)\,ds$ $\forall t \in [0, T]$, $c \ge 0$, $L \ge 0$.

Then $\boxed{r(t) \le c \exp\{Lt\}}$ $\forall t \in [0, T]$

(we took here $t_0 = 0$)

it is a simple variant of Gronwall's inequality.

Proof of the Lemma.

Let $v(t) = C + \int_0^t L\, r(s)\,ds > 0$

then $v'(t) = L\, r(t)$; $r(t) \leq v(t)$

and $\dfrac{v'(t)}{v(t)} = \dfrac{L\, r(t)}{v(t)} \leq L$

$$\frac{d}{dt}\left(\log v(t)\right) \leq L$$

$$\log v(t) \leq \log v(0) + Lt$$

$$v(0) = C; \quad \Rightarrow \quad v(t) \leq C \cdot e^{Lt}$$

$$\Rightarrow \quad r(t) \leq C \cdot e^{Lt} \quad \blacksquare$$

If $C = 0$ we can apply the same argument for $C_i > 0$, $C_i \xrightarrow[i \to \infty]{} 0$ and get the same result.

Proof of the general case uses method with integrating factor. We take again without loss of generality $t_0 = 0$.

$$r(t) \leq r(0) + \int_0^t L\, r(s) + M\,ds =$$

$$= r(0) + L \int_0^t r(s)\,ds + tM$$

Let $R(t) = \int_0^t r(s)\,ds$

$R'(t) = r(t)$; $R'(t) \le r(0) + R(t)L + M\cdot t$

$R'(t) - LR(t) \le r(0) + Mt$ | $\times e^{-Lt}$ multiply

$\left(e^{-Lt} R(t)\right)' \le r(0)\, e^{-L(t)} + M\cdot(t)\, e^{-Lt}$ integrate

$$e^{-Lt}R(t) \le r_0 \int_0^t e^{-Ls}\, ds + \int_0^t M s\, e^{-Ls}\, ds =$$

$$= \frac{r_0}{L}\left(1 - e^{-Lt}\right) - \frac{M}{L^2}\left(e^{-Lt} + L - 0\right) +$$

$$+ \int^{tL} \frac{M}{L^2} e^{-\tau} d\tau = \frac{r_0}{L}\left(1 - e^{-Lt}\right) - \frac{M}{L^2}\left(e^{-Lt} Lt - 0\right)$$

$$+ \frac{M}{L^2}\left(1 - e^{-Lt}\right)$$

Therefore

$$R(t) \le \frac{r_0}{L}\left(e^{Lt} - 1\right) - \frac{M}{L^2}\left(Lt + 1\right) + \frac{M}{L^2} e^{Lt}$$

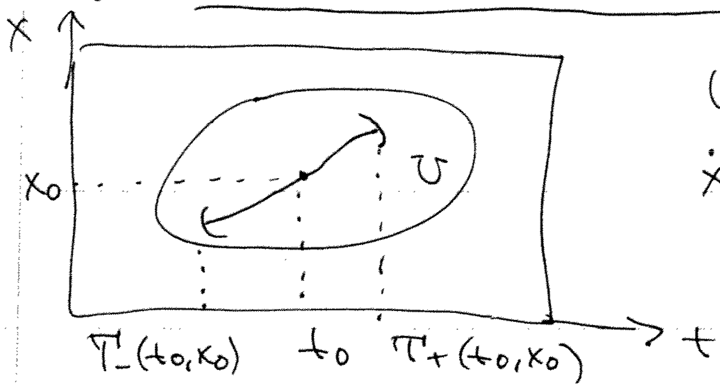
and

$$r(t) \le \cancel{r_0} + r_0\left(e^{Lt} \cancel{-1}\right) - \frac{M}{L}\left(\cancel{Lt} + 1\right)$$

$$+ \frac{M}{L} e^{Lt} + \cancel{Mt} =$$

$$= r_0 e^{Lt} + \frac{M}{L}\left(e^{Lt} - 1\right)$$

§26. Extensibility of solutions.



Let solutions to $\dot{x} = f(t,x)$ $\exists$ and unique locally. $f \in C^1(U, R^{n+1})$

One is interested in maximal time interval $(T_-(t_0,x_0), T_+(t_0,x_0))$ on which a local solution going throung (t_0,x_0) can be extended. Such a solution on (T_-, T_+) is called maximal solution.

$x(t)$ is called global solution if it is defiened for $t \in R^1$.

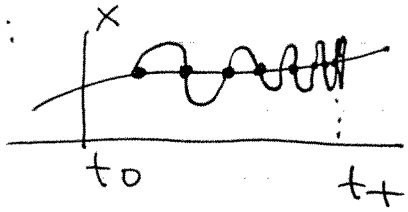
Lemma 2.14 Let $\phi(t)$ be a solution to $\dot{\phi} = f(t, \phi(t))$, $\phi(t_0) = x_0$ on the interval (t_-, t_+), including t_0. An extension to the interval $(t_-, t_+ + \varepsilon)$ for some $\varepsilon > 0$ exists if and only if there is a sequence of time points $t_m \in (t_-, t_+)$ such that $t_m \underset{m\to\infty}{\longrightarrow} t_+$ and

$$\lim_{m\to\infty} (t_m, \phi(t_m)) = (t_+, y) \in U$$

(U - domain for f)

A similar statement is valid for the extension to $(t_- - \varepsilon, t_+)$

Proof. Only direct statement is non-trivial.
We show first that $\lim_{t \uparrow t_+} \phi(t) = y$,
namely that the left limit exists and coinsides with the limit along the sequence $\{t_m\}$. It doesnot be correct: $\phi(t)$ can oscillate around t_+:
like $\sin\left(\frac{1}{t-t_+}\right)$



Intuitive argument against it is that $f(t, x(t)) = \dot{x}$ is continuous and bounded on a closed interval. Now formal proof comes.
V is opened $\Rightarrow \exists$ $[t_+ - \delta, t_+] \times B_\delta(y) \subset V$ for a small δ. Choosing $m > N$ large enough we get $t_m \in (t_+ - \varepsilon, t_+)$ and $\phi(t_m) \in B_\delta(y)$. Let $\sup_{(t,x) \in \bar{V}} f(t,x) = M$.
Taking a subsequence we can have t_m - monotone. ($\bar{V}$ is a compact)
suppose the limit $\lim_{t \uparrow t_+} \phi(t) = y$ does not exist.
Then there must be another equence $\{\tau_m\}_{m=1}^{\infty}$, $\tau_m \to t_+$ and monotone such that $|\phi(\tau_m) - y| \geq \gamma > 0$
We choos $\gamma < \delta$ and $\tau_m \geq t_m$ which is always possible by taking a subsequence in $\{\tau_m\}$. Then

$$0 < \gamma < |\phi(\tau_m) - y| \leq |\phi(t_m) - \phi(\tau_m)| + |\phi(t_m) - y| \leq M(\tau_m - t_m) + |\phi(t_m) - y|$$

$\tau_m - t_m \xrightarrow[m \to \infty]{} 0$, $|\phi(t_m) - y| \xrightarrow[m \to \infty]{} 0$. Contradiction!

Therefore $\lim_{t \uparrow t_+} \Phi(t) = y$.

Now we can consider a solution $\hat{\Phi}(t)$ to $\dot{x} = f(t,x)$ with initial condition $x(t_+) = y$. We can glue $\phi(t)$ and $\hat{\Phi}(t)$ together at t_+ to obtain a solution on an extended interval $(t_-, t_+ + \epsilon)$. It is a continuous function by construction (left and right limits are the same in $t = t_+$). The same is valid for derivatives: left and right derivatives are equal to $f(t_+, y)$ => therefore this composite function is differentiable at t_+ and is a solution to $\dot{x} = f(t,x)$ defined on $(t_-, t_+ + \varepsilon)$. ∎

Corollary 2.15 (Weaker limitations for extension.)

Let $\phi(t)$ be a solution $\dot{x} = f(t,x)$ on the interval (t_-, t_+). Suppose $\exists$ compact C in $\mathbb{R}^n$ such that $[t_0, t_+] \times C \subset U$ such that $\exists$ a sequence $\{t_m\} \in [t_0, t_+)$ converging to t_+ such that $\Phi(t_m) \in C$.

Then there is an extension of $\Phi(t)$ to some interval $(t_-, t_+ + \varepsilon)$ for an $\varepsilon > 0$.

In particulal if $\forall\, t_+ > t_0$ there is a compact C (depending on t_+) with such properties, then the solution $\phi(t)$ can be extended to all $t > t_0$.

Proof of 2.15. $\{\phi(t_m)\} \subset C$ - compact then there must be a convergent subsequence $\{\phi(t_{m_k})\}$

$$\phi(t_{m_k}) \xrightarrow[k\to\infty]{} y \quad , \; y \in C$$

By Lemma 2.14. the solution $\phi(t)$ must have an extension. $\boxtimes$

If there is such $C(t_+)$ for any $t_+ > t_0$, we see that $\phi(t)$ can be extended past any time t_+. $\boxtimes$

A useful negation to the Lemma 2.15 is the following.

Lemma 2.16.

Let $I(t_0, x_0) = \big(T_-(t_0,x_0),\, T_+(t_0,x_0)\big)$ be a maximal existence interval for a solution starting at (t_0, x_0).

If $T_+(t_0,x_0) < \infty$ then the solution $\phi(t)$ must leave any compact C such that $[t_0, T_+] \times C \subset U$ as t approaches T_+. In particular if $U = \mathbb{R}^{n+1}$ the solution must tend to infinity as $t \to T_+$

<u>Theorem 2.17.</u> Suppose $U = R \times R^n$ and for every $T > 0$ $\exists M(T), L(T)$ such that

$$|f(x,t)| \leq M(T) + L(T)|x|, \quad (t,x) \in [-T,T] \times R^n$$

Then all solutions to $\dot{x} = f(t,x)$, $x(t_0) = x_0$ are defined for all $t \in R$.

Shortly we say that if $f(t,x)$ grows not faster than linearly in x, then all solutions are global.

<u>Proof.</u> We take $t_0 = 0$ for shorter writing.

$$\phi(t) = \phi(0) + \int_0^t f(s, \phi(s))\, ds$$

$$|\phi(t)| \leq |\phi(0)| + \int_0^t M + L(\phi(s))\, ds$$

$t \in [0, T]$

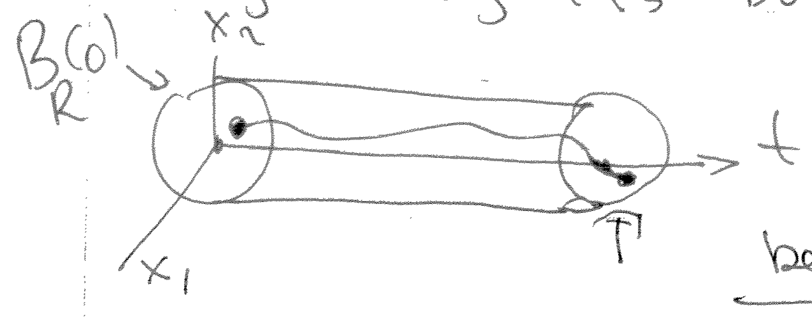
Gronwall inequality implies:

$$|\phi(t)| \leq |x_0|\, e^{LT} + \frac{M}{L}\left(e^{LT} - 1\right) = R$$

Therefore $\phi(t) \in \overline{B_R(0)}$ and can be <u>extended</u> to a larger time interval.

$\overline{[0,T] \times B_R(0)}$ is a compact in the domain of f.

Another argument is that $\underline{\phi(t)}$ can escape the cylinder $[0,T] \times B_R(0)$ only throug its bottom at $t = T$



because of the estimate above, therefore it can <u>be extended for any time</u>