Extra Notes 6 (24/5)

Definition E.27. Let $G = (V, E)$ be a graph. A subset $W \subseteq V$ is said to be a *vertex cover* for G if, for every edge $\{x, y\} \in E$, either $x \in W$ or $y \in W$ (or both).

Theorem E.28. (König's theorem) *Let $G = (X, Y, E)$ be a bipartite graph. Then the maximum size of a matching in G equals the minimum number of vertices in a cover of G.*

Proof: If M is any matching and W is any cover, then it is clear that $|M| \leq |W|$ since
- W a cover $\Rightarrow$ every edge of M matches either one or two vertices of W
- M a matching $\Rightarrow$ no element of W is matched twice.

It therefore suffices to prove the existence of a matching M and a cover W such that $|M| = |W|$. To do so, we consider the network $\vec{G} = (V, \vec{E})$ where

- $V = X \sqcup Y \sqcup \{s, t\}$,
- $\vec{E} = \{(x, y) : \{x, y\} \in E\} \sqcup \{(s, x) : x \in X\} \sqcup \{(y, t) : y \in Y\}$,
- $c(\vec{e}) = 1 \; \forall \, \vec{e} \in \vec{E}$.

We apply the Ford-Fulkerson algorithm to $\vec{G}$, starting from the everywhere-zero flow. Considering how the algorithm works, we can without further comment make the following observations:

- At every stage of the procedure, the flow along every edge will be either zero or one - so each edge is always either empty or saturated.
- Each time an f-augmenting path is found, the total flow increases by one. Exactly one extra edge from s to X becomes saturated, plus exactly one extra edge from Y to t.
- At every stage of the procedure, the saturated edges between X and Y form a matching M in G (by conservation of flow since, for every $x \in X$ only one arc enters x, namely that from s and similarly, for every $y \in Y$ only one arc exits y, namely that to t).
- Each f-augmenting path consists of three parts: (i) an edge from s to some $x \in X$, (ii) an M-augmenting path from x to some $y \in Y$ (iii) an edge from y to t.
- At every step, an f-augmenting path exists if and only if an M-augmenting path exists.

It follows from these observations and Proposition 19.1 that the strength of the final flow f_∞ equals the maximum size of a matching in G. Let (S, T) be the corresponding cut. We know by Theorem 20.5 that $|f_\infty| = c(S, T)$, so it just remains to produce a vertex cover W for G such that $|W| = c(S, T)$.

A priori we have $s \in S$, $t \in T$ and partitions[1] $X = A \sqcup B$, $Y = C \sqcup D$ such that $S = \{s\} \cup A \cup C$ and $T = \{t\} \cup B \cup D$. The arcs contributing to the capacity of the

[1] The partitions need not be *proper*, i.e.: the four sets A, B, C, D need not all be non-empty.

cut are those from s to B, those from C to t and those from A to D. Hence,

$$c(S,\, T) = |B| + |C| + |E(A,\, D)|, \tag{0.30}$$

where $E(A,\, D)$ denotes the collection of arcs from A to D. Now $A \subseteq S$, which means every vertex in A is reachable by an f_∞-augmenting path from s. On the other hand, no vertex in D is reachable. This means that every arc from A to D must be saturated. But, as already noted in the 3rd observation above, for any $a \in A$, there is only one arc in $\vec{G}$ entering a, namely the arc $(s,\, a)$. Hence, by flow conservation, at most one saturated arc can exit a. It follows that the edges in $E(A,\, D)$ must form a matching and if $A_1 \subseteq A$ is the set of matched vertices in A then $|E(A,\, D)| = |A_1|$. Substituting into (0.30) we have

$$c(S,\, T) = |B| + |C| + |A_1|. \tag{0.31}$$

Now take $W = B \cup C \cup A_1$. If $\{x,\, y\} \in E(G)$ then either

- $x \in B$, or
- $y \in C$, or
- $x \in A$ and $y \in D$, in which case $x \in A_1$.

Hence W is a vertex cover for G, of size equal to $c(S,\, T)$, v.s.v.