Fourier analysis (MMG710/TMA362)

Time: 2009-08-19, 08.30–13.30
Tools: No calculator or handbook is allowed
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Grades: For MMG710 grades are G (12-17 points) and VG (18-24 points). For TMA362 grades are 3 (12-14 points), 4 (15-17 points) and 5 (18-24 points).

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1 Find a solution to the boundary value problem

$$\begin{cases} u'_t = 4u''_{xx}, & t > 0, \quad 0 < x < \pi, \\ u(0,t) = u(\pi,t) = 0, & t > 0, \\ u(x,0) = \sin x + 2\sin(2x). \end{cases}$$

(3p)

2 Using that the Fourier transform of $f(x) = e^{-|x|}$ is $\hat{f}(\xi) = 2/(\xi^2+1)$, find the Fourier transform of $x/(x^2+1)^2$. (3p)

3 (a) Expand the function $f(x) = \sin(|x|)$ as a Fourier series on the interval $|x| < \pi$. (3p)

(b) Use the result of (a) to compute

$$\sum_{k=1}^{\infty} \frac{1}{(4k^2-1)^2}.$$

(3p)

4 Find a function with Laplace transform $(1-e^{-s})^2/s^2$. Sketch the graph of the function. (4p)

5 Formulate and prove Bessel's inequality, either for Fourier series or in a more general setting. (4p)

6 Let $L(f) = rf'' + r'f' + pf$ be a Sturm-Liouville operator. Consider an eigenvalue problem problem of the form

$$L(f) = \lambda f, \qquad f(0) = f(1) = 0.$$

(a) Prove that the problem is self-adjoint in the sense that $\langle L(f), g\rangle = \langle f, L(g)\rangle$, where $\langle\cdot,\cdot\rangle$ is a certain inner product, and where f and g are sufficiently differentiable functions satisfying the boundary conditions. (2p)

(b) Using part (a), prove that all eigenvalues λ are real. (2p)

Good luck! Hjalmar