Chalmers tekniska högskola och Göteborgs universitet
Matematik
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Proof of Bessel's inequality

We write the proof of Bessel's inequality in a way that makes the analogy with linear algebra more transparent. In Folland's book, this is done in Chapter 3.3, in a more general context.

Recall that Bessel's inequality is as follows.

Theorem: If f is 2π-periodic and Riemann integrable, with Fourier coefficients c_n, then

$$\sum_{-\infty}^{\infty} |c_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx.$$

For the proof, we use the notation

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{g(x)} \, dx,$$

$$\|f\| = \sqrt{\langle f, f \rangle} = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx \right)^{\frac{1}{2}}.$$

This scalar product satisfies some obvious properties. For instance, it is conjugate-symmetric:

$$\langle g, f \rangle = \overline{\langle f, g \rangle} \tag{1}$$

We need the following counterpart of Pythagoras' theorem.

Lemma: If $\langle f, g \rangle = 0$, then

$$\|f + g\|^2 = \|f\|^2 + \|g\|^2.$$

Proof: Using some properties, which the reader should check,

$$\|f + g\|^2 = \langle f + g, f + g \rangle = \langle f, f \rangle + \langle f, g \rangle + \langle g, f \rangle + \langle g, g \rangle.$$

Since $\langle f, g \rangle = 0$ and, by (1), $\langle g, f \rangle = 0$, this equals

$$\langle f, f \rangle + \langle g, g \rangle = \|f\|^2 + \|g\|^2.$$

We now introduce the functions

$$e_n(x) = e^{inx}.$$

Since, for real x, $\overline{e^{inx}} = e^{-inx}$, these are orthonormal in the sense that

$$\langle e_k, e_l \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ikx} e^{-ilx} \, dx = \begin{cases} 1, & k = l, \\ 0, & k \neq l \end{cases}$$

(we used this integral before!). The Fourier coefficients can be written

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \langle f, e_n\rangle.$$

Thus, Bessel's inequality takes the form

$$\sum_{-\infty}^{\infty} |\langle f, e_n\rangle|^2 \le \|f\|^2. \tag{2}$$

Let us now look at the function

$$u(x) = \sum_{n=-N}^{N} \langle f, e_n\rangle e_n(x).$$

This might remind you of a formula for orthogonal projections from linear algebra. In fact, u can be viewed as the orthogonal projection of f to the space spanned by $e_{-N}, \dots, e_N$. To give a precise meaning to this, note that if $-N \le k \le N$, then

$$\langle u, e_k\rangle = \sum_{n=-N}^{N} \langle f, e_n\rangle\langle e_n, e_k\rangle = \langle f, e_k\rangle,$$

or equivalently

$$\langle f - u, e_k\rangle = 0.$$

By linearity, it follows that

$$\langle f - u, u\rangle = 0.$$

Pythagoras' theorem, applied to the decomposition $f = u + (f - u)$, then gives

$$\|f\|^2 = \|u\|^2 + \|f - u\|^2. \tag{3}$$

Another application of Pythagoras' theorem (or, rather, its extension to a sum with several terms) gives

$$\|u\|^2 = \sum_{n=-N}^{N} |\langle f, e_n\rangle|^2 \tag{4}$$

Combining this with (3), we see that

$$\sum_{n=-N}^{N} |\langle f, e_n\rangle|^2 \le \|f\|^2.$$

We can now let $N \to \infty$ to deduce Bessel's inequality (2).

Note also that, if we believe Fourier's intuition that "any" function can be written as a Fourier series, u should approach f as N grows, so (4) should tend to $\|f\|^2$. This suggests that we actually have equality in Bessel's inequality! This turns out to be always true, but it has to be proved later (Folland, Theorem 3.4).