Fourier analysis. Additional exercises.

1. (a) Consider the wave equation $u''_{tt} = c^2 u''_{xx}$. Show that a solution defined in all of $\mathbb{R}^2$ can be written $u(x,t) = f(x+ct) + g(x-ct)$, that is, as the sum of two waves travelling along the x-axis with speed c in opposite directions.

   (b) Show that the standing wave $u(x,t) = \sin(x)\sin(ct)$ is a solution of the wave equation. What does it *look* like? Decompose it as a sum of two waves as in (a).

   (c) Find a solution to the wave equation $u''_{tt} = 4u''_{xx}$, valid for all x and t, such that $u(x,0) = x^2$, $u'_t(x,0) = x$.

2. Consider the vibrating string with fixed endpoints on $0 \le x \le \pi$. Supppose that the initial conditions are $u(x,0) = \sin(2x)$, $u'_t(x,0) = 3\sin(5x)$. What is the solution $u(x,t)$?

3. If you know the Fourier coefficients of f, what can you say about the Fourier coefficients of $f(x-a)$ and of $e^{ikx}f(x)$ (where a is real and k is integer)? Compare with the shift rules for Laplace transform.

4. When f and g are 2π-periodic Riemann integrable functions, define their convolution by

$$(f * g)(x) = \frac{1}{2\pi}\int_0^{2\pi} f(y)g(x-y)\,dy.$$

   Denoting Fourier coefficients by $c_n(f)$, show that $c_n(f*g) = c_n(f)c_n(g)$.

5. Let f be the 2π-periodic function defined by $f(x) = e^{\cos(x^2)}$ for $0 \le x < 2\pi$. What is the value of its Fourier series at $x = 4\pi$?

6. Let f be the 2π-periodic function defined by $f(x) = \cosh(x) = (e^x + e^{-x})/2$ for $|x| \le \pi$. Express it as a Fourier series. Compute

$$\sum_{k=0}^{\infty} \frac{1}{k^2+1}.$$

7. Let f be the 2π-periodic function defined by $f(x) = \cos(ax)$ for $|x| \le \pi$, where a is not an integer. Express it as a Fourier series. Deduce the identity

$$\pi\cot(\pi a) = \lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1}{a+n}, \qquad a \notin \mathbb{Z}.$$

8. Solve the heat conduction problem

$$\begin{cases} u'_t = 3u''_{xx}, & 0 \le x \le \pi, \quad t > 0, \\ u(0,t) = u(\pi,t) = 0, & t \ge 0, \\ u(x,0) = \sin x\cos(4x), & 0 \le x \le \pi. \end{cases}$$

9. Solve the heat conduction problem

$$\begin{cases} u'_t = u''_{xx}, & 0 \le x \le \frac{\pi}{2}, \quad t > 0, \\ u(0,t) = u'_x(\pi/2,t) = 0, & t \ge 0, \\ u(x,0) = x(\pi - x), & 0 \le x \le \frac{\pi}{2} \end{cases}$$

10. A large steak with thickness l and heat diffusivity a is to be cooked in an oven with temperature T. It is taken directly from the fridge, where the temperature is 0. The temperature inside the steak can then be described by

$$\begin{cases} u'_t = a u''_{xx}, & 0 \le x \le l, \quad t > 0, \\ u(0,t) = u(l,t) = T, & t > 0, \\ u(x,0) = 0, & 0 \le x \le l. \end{cases}$$

   (a) Determine, as a series, $u(x,t)$. Since the boundary conditions are inhomogeneous, one should shift the temperature scale so that T corresponds to 0 (equivalently, introduce the function $v(x,t) = u(x,t) - T$).

   (b) Suppose all terms in the series except the first can be ignored. When will the steak be cooked if that happens when the temperature is everywhere at least $T/4$? How much longer does it take if the thickness of the steak is doubled?

11. Addendum to Folland, Exercise 2.5.5: Where should the string be plucked if we do not wish to hear the sixth overtone (corresponding to the seventh term in the Fourier series)? Motivation: The first five overtones are close to tones in the usual scale. The sixth overtone is somewhere between two half-tones, and thus sounds false (at least to people accustomed to European music).

12. Addendum to Folland, Exercise 2.5.6: Can we choose δ so the sixth overtone is avoided?

13. Find a 2π-periodic solution to the equation

$$x'' + 2x' + x = \cos(t).$$

Try to do this in several different ways! Also find the solution to

$$x'' + 2x' + x = \cos(t), \qquad x(0) = x'(0) = 0$$

and observe that it approaches the periodic solution as t grows.

14. Find a 1-periodic solution to the equation

$$x'' + 2x' + x = \{t\},$$

where $\{t\}$ is the fractional part of t (that is, the 1-periodic function defined by t for $0 \le t < 1$). Give the answer in real form.

15. Show that, in an inner product space, if $u_n \to u$ and $v_n \to v$ (in norm), then $\langle u_n, v_n \rangle \to \langle u, v \rangle$.

16. Show that, if $(e_k)_{k=1}^{\infty}$ is a complete orthonormal system in an inner product space, then

$$\langle f, g \rangle = \sum_{k=1}^{\infty} \langle f, e_k \rangle \langle e_k, g \rangle.$$

17. Find an orthonormal basis for the space of first degree polynomials, with inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x)\, dx.$$

18. Use the previous exercise to determine the constants a, b that minimize the integral

$$\int_{-1}^{1} |e^x - ax - b|^2\, dx.$$

19. Apply Parseval's formula to the 2π-periodic function $f(x) = x$, $|x| < \pi$. Use the result to compute $\sum_1^\infty \frac{1}{n^2}$.

20. Apply Parseval's formula to the 2π-periodic function $f(x) = x^2$, $|x| < \pi$. Use the result to compute $\sum_1^\infty \frac{1}{n^4}$.

21. Define $J_n(x)$ through the Fourier series

$$e^{ix \sin(t)} = \sum_{n=-\infty}^{\infty} J_n(x) e^{int}$$

(they are called Bessel functions). Compute, for $x \in \mathbb{R}$,

$$\sum_{n=-\infty}^{\infty} |J_n(x)|^2.$$

22. Compute the Fourier transform of

(a) $e^{-|x|}$,

(b) $x\, \chi_{[-1,1]}(x)$,

(c) $\sin x\, \chi_{[-\pi,\pi]}(x)$,

(d) $e^{-x} H(x)$,

(e) $e^{-|x|} \cos(x)$.

Here, H is Heaviside's function and $\chi_{[a,b]}(x)$ is the characteristic function of $[a, b]$, that is, $\chi_{[a,b]}(x) = 1$ for $x \in [a, b]$ and 0 else.

23. If $f(x)$ has Fourier transform $\hat{f}(\xi)$, what is then the Fourier transform of $\cos(x)f(2x+1)$?

24. Complete the proof of (a) and (b) in Folland, Theorem 7.5.

25. Using that $1/(x^2+1)$ has Fourier transform $\pi e^{-|\xi|}$, find the Fourier transforms of

    (a) $\dfrac{1}{x^2+6x+13}$,

    (b) $\dfrac{x}{(x^2+1)^2}$.

26. Use Fourier transform to find a function f such that

$$\int_{-\infty}^{\infty} f(x-y)e^{-|y|}\,dy = 2e^{-|x|} - e^{-2|x|}.$$

27. For a and b positive, use Fourier transform to compute the integrals

    (a) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(at)\sin(b(u-t))}{t(u-t)}\,dt$,

    (b) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(at)\sin(bt)}{t^2}\,dt$.

28. Using Exercise 22(e), compute

$$\int_{-\infty}^{\infty} \frac{(x^2+2)^2}{(x^4+4)^2}\,dx.$$

29. Find (as an integral) a solution to the heat equation $u_{xx} = u_t$ for $t > 0$, $x \in \mathbb{R}$, where $u(x,0) = 1$ for $|x| < 1$ and 0 else.

30. Show that, if f is even, then under suitable conditions

$$f(x) = \frac{2}{\pi}\int_0^{\infty} F(\xi)\cos(\xi x)\,d\xi,$$

where

$$F(\xi) = \int_0^{\infty} f(x)\cos(\xi x)\,dx.$$

Also give a corresponding formula for odd functions.

Answers and hints

1. (a) **Hint:** Make the change of variables $y = x + ct$, $z = x - ct$.
(c) $x^2 + 4t^2 + xt$.

2. $u(x,t) = \sin(2x)\cos(2ct) + \frac{3}{5c}\sin(5x)\sin(5ct)$.

5. $\dfrac{e^{\cos(4\pi^2)} + e}{2}$.

6. The Fourier series is
$$\frac{\sinh(\pi)}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n e^{inx}}{1+n^2}$$
and the sum is
$$\frac{\pi}{2}\coth(\pi) - \frac{1}{2}.$$

7. The Fourier series is
$$\frac{a\sin(\pi a)}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n e^{inx}}{a^2-n^2}.$$

8. $\frac{1}{2}(e^{-75t}\sin(5x) - e^{-27t}\sin(3x))$.

9. $\frac{8}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}e^{-(2k+1)^2 t}\sin((2k+1)x)$.

10. (a) $T - \frac{4T}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}e^{-(2k+1)^2\pi^2 at/l^2}\sin((2k+1)\pi x/l)$.
(b) The cooking time is approximately $\frac{l^2}{a\pi^2}\ln\left(\frac{16}{3\pi}\right)$, so a twice as thick steak takes four times as long to cook.

11. Second question: In one of the points $a = kl/7$, $k = 1, 2, \dots, 6$. Note that these are the nodes of the sixth overtone.

12. Second question: Yes, take $\delta = kl/7$, $k = 1, 2, \dots, 6$.

13. $x(t) = \frac{1}{2}\sin t$, $x(t) = \frac{1}{2}\sin t - \frac{1}{2}te^{-t}$.

14. $x(t) = \frac{1}{2} - \sum_{n=1}^{\infty}\frac{\sin(2\pi nt - 2\arctan(\pi n))}{\pi n(1+4\pi^2 n^2)}$.

15. **Hint:** Write $\langle u, v\rangle - \langle u_n, v_n\rangle = \langle u, v - v_n\rangle + \langle u - u_n, v_n\rangle$.

16. **Hint:** Use Exercise 15.

17. For instance, $\frac{1}{\sqrt{2}}$ and $\sqrt{\frac{3}{2}}x$.

18. $a = 3e^{-1}$, $b = (e - e^{-1})/2$.

20. $\dfrac{\pi^4}{90}$.

21. 1, for any x.

22. (a) $\dfrac{2}{\xi^2+1}$, (b) $2i\dfrac{\xi\cos\xi - \sin\xi}{\xi^2}$, (c) $\dfrac{2i\sin(\pi\xi)}{\xi^2-1}$, (d) $\dfrac{1}{1+i\xi}$, (e) $\dfrac{2(\xi^2+2)}{\xi^4+4}$.

23. $\dfrac{1}{4}\left(e^{\frac{1}{2}i(\xi-1)}\hat{f}\left(\dfrac{\xi-1}{2}\right) + e^{\frac{1}{2}i(\xi+1)}\hat{f}\left(\dfrac{\xi+1}{2}\right)\right)$.

25. (a) $\dfrac{1}{2}\pi e^{3i\xi-2|\xi|}$, (b) $-\dfrac{1}{2}\pi i\xi e^{-|\xi|}$.

26. $f(x) = \dfrac{3}{2}e^{-2|x|}$.

27. (a) $\pi\dfrac{\sin(cu)}{u}$, (b) πc, where $c = \min(a, b)$.

28. $3\pi/8$.

29. $u(x,t) = \dfrac{1}{\sqrt{4\pi t}}\displaystyle\int_{-1}^{1} e^{-(x-y)^2/4t}\,dy$.

30. See Folland, page 238.