Chalmers tekniska högskola och Göteborgs universitet
Matematik
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Butterworth filters

In signal analysis, one often assumes that the "message" part of the signal is band-limited, that is, it only contain frequencies in some interval $|\omega| \leq C$. On the other hand, signals may be disturbed by noise, which has a large part of its spectrum in high frequencies. Therefore, it is useful to have a device that, as far as possible, filters out high frequencies while preserving low frequencies. Such *low-pass filters* can be found in TVs, cell phones, image processing programs etc.

We consider filters that can be modelled by an ODE

$$\begin{cases} a_n x^{(n)} + \cdots + a_0 x = u, \\ x(0) = x'(0) = \cdots = x^{(n-1)}(0) = 0, \end{cases} \tag{1}$$

where u is the in-signal and x the (filtered) out-signal.

As we have seen, the Laplace transforms of the in- and out-signal are related by

$$X(i\omega) = G(i\omega)U(i\omega),$$

where

$$G(s) = \frac{1}{a_n s^n + \cdots + a_0}. \tag{2}$$

We write the variable as $i\omega$, since ω can be interpreted as frequency (recall that $X(i\omega) = \hat{x}(\omega)$ for causal functions).

In an *ideal* low-pass filter, frequencies with $|\omega| > c$ are turned off, while smaller frequencies are preserved except for possibly a phase-shift. This corresponds to the choice

$$|G(i\omega)| = \begin{cases} 1, & |\omega| < c, \\ 0, & |\omega| > c. \end{cases}$$

However, this is impossible if G has the form (2). Thus, in practice one must work with approximations to the ideal filter.

One commonly used approximation is Butterworth filters, which have

$$|G(i\omega)| = \frac{1}{\sqrt{1 + (\omega/c)^{2n}}}, \qquad n = 1, 2, 3, \dots. \tag{3}$$

This is a good approxoimation to the ideal filter, especially if n is large. Let us show that (3) can be achieved with G of the form (2). That is, we want to find a polynomial

$$p(s) = a_n s^n + \cdots + a_0$$

with

$$|p(i\omega)|^2 = 1 + (\omega/c)^{2n}.$$

Since p should have real coefficients, $|p(i\omega)|^2 = p(i\omega)p(-i\omega)$, so equivalently

$$p(s)p(-s) = 1 + (s/ic)^{2n}. \tag{4}$$

Consider the zeroes of $1 + (s/ic)^{2n}$. Note that if s is a zero, then $-s$ is a zero. Moreover, there is no zero at the imaginary axis, since if $s = it$, then $1 + (s/ic)^{2n} = 1 + (t/c)^{2n} > 0$ for t and c real. Thus, there are exactly n zeroes in the left halfplane. We call these $s_1, \dots, s_n$. Then,

$$\begin{aligned} 1 + (s/ic)^{2n} &= \frac{1}{(ic)^{2n}}(s - s_1)\cdots(s - s_n)(s + s_1)\cdots(s + s_n) \\ &= \frac{1}{c^{2n}}(s - s_1)\cdots(s - s_n)(-s - s_1)\cdots(-s - s_n). \end{aligned}$$

We see that (4) holds with

$$p(s) = \frac{1}{c^n}(s - s_1)\cdots(s - s_n).$$

The corresponding system (1) is then a Butterworth filter. It is not hard to explain how to build it using standard electronic components, but we do not do so here.

Consider the case $n = 2$. The equation $1 + (s/c)^4 = 0$ has zeroes

$$\frac{c}{\sqrt{2}}(\pm 1 \pm i)$$

(all four sign choices). Thus,

$$p(s) = \frac{1}{c^2}\left(s - \frac{c}{\sqrt{2}}(-1 + i)\right)\left(s - \frac{c}{\sqrt{2}}(-1 - i)\right) = \frac{1}{c^2}\left(s^2 + \sqrt{2}cs + c^2\right).$$

Alternatively, write

$$1 + \frac{s^4}{c^4} = \frac{1}{c^4}\left((s^2 + c^2)^2 - 2c^2 s^2\right) = \frac{s^2 + \sqrt{2}cs + c^2}{c^2} \cdot \frac{s^2 - \sqrt{2}cs + c^2}{c^2},$$

which gives (4) directly. We conclude that the second order Butterworth filter is described by the system

$$\frac{1}{c^2}(x'' + \sqrt{2}cx' + c^2 x) = u.$$

As an example, choose $c = 4$ and input

$$u(t) = 9.5\sin(t) - \sin(9.5t).$$

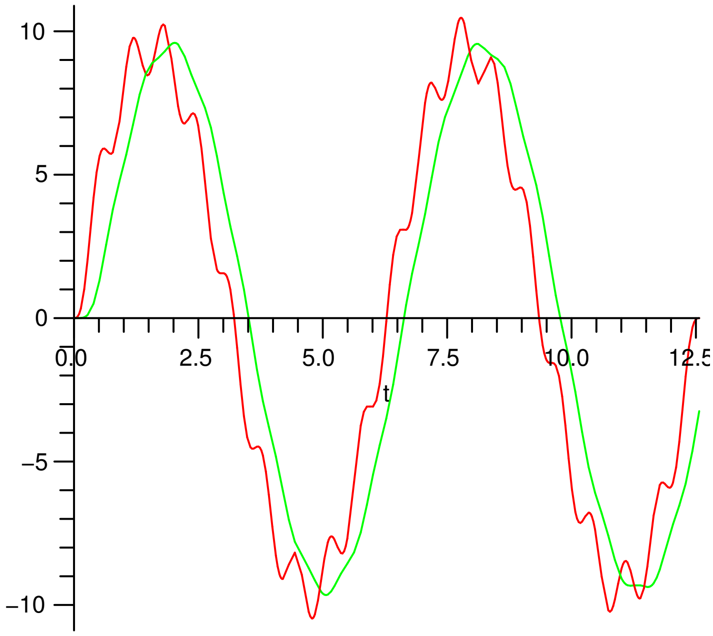
The first term has frequency $1 < 4$ and should be preserved (up to a phase-shift). The second term has frequency $9.5 > 4$ and should be filtered out. The output is the solution to

$$\frac{1}{16}x'' + \frac{\sqrt{2}}{4}x' + x = 9.5\sin(t) - \sin(9.5t), \qquad x(0) = x'(0) = 0.$$

After rounding off constants, the solution can be written

$$x(t) = 9.48\sin(t - 0.36) + 0.17\sin(9.5t + 0.63) + 3.26e^{-2.83t}\sin(2.83t + 1.68).$$

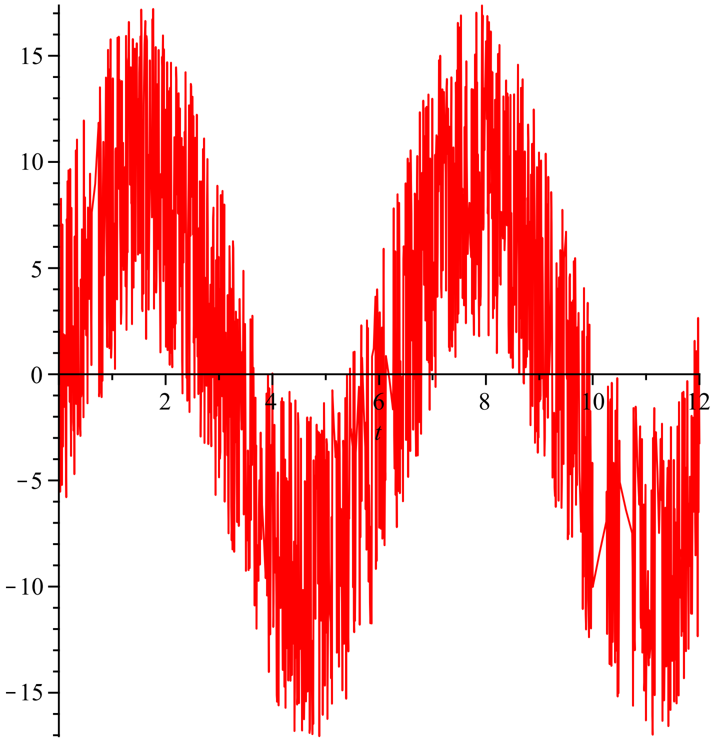
Note how the first term is essentially just phase-shifted, while the second term is damped. The third term (the transient) is negligible except for very small t (this is why we chose p so that all zeroes are in the left-half plane). Plotting u and x together, the effect of the filter is seen very clearly.



We also consider an example where the same signal is disturbed by a random noise. More precisely, we take as input a step function

$$u(t) = 9.5\sin\left(\frac{i}{1000}\right) + a_i, \qquad \frac{i}{1000} < t < \frac{i+1}{1000},$$

where a_i are numbers chosen uniformly at random in the interval $[-8, 8]$. In the following pictures, we plot first the in-signal $u(t)$, and then the out-signal $x(t)$.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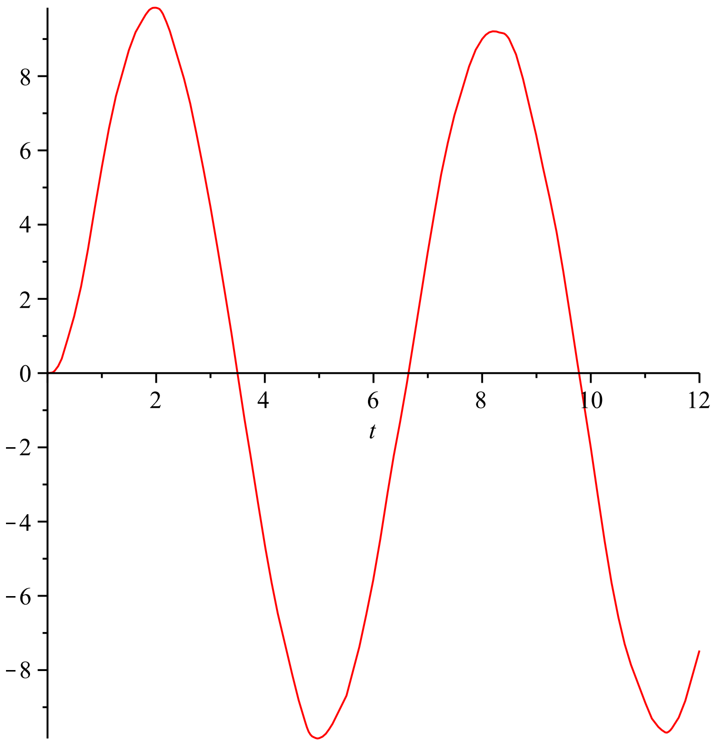
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