Chalmers tekniska högskola och Göteborgs universitet
Matematik
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Gibbs phenomenon

We have seen (Folland, Thm. 2.5) that if f is 2π-periodic, piecewise C^1 and continuous, then its Fourier series converges *uniformly* on $\mathbb{R}$. If, on the other hand, f is *not* continuous, convergence cannot be uniform. This follows from the fact that if a sequence of continuous functions (e.g. the partial sums $\sum_{-N}^{N} c_n e^{inx}$) converges uniformly, then the limit function is continuous. It turns out that for Fourier series the situation is even worse. The partial sums develop "spikes" close to each jump point, whose heights remain positive as $N \to \infty$ (the limit height of the largest spike is about 9% of the height of the jump). This fact is known as the Gibbs phenomenon. See Folland, Figure 2.8 for an illustration.

We will first illustrate the Gibbs phenomenon by an explicit example, and then explain how the general case actually follows from that example. Consider the 2π-periodic function s defined by $s(x) = \pi - x$ for $0 < x < 2\pi$. It has a jump of height 2π at $x = 0$. Its Fourier series is

$$2\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}.$$

Thus, the error in the Nth Fourier approximation, for $0 < x < \pi$, is

$$g_N(x) = 2\sum_{n=1}^{N} \frac{\sin(nx)}{n} - (\pi - x).$$

Since we are interested in the maximum error, we compute the derivative

$$g_N'(x) = 1 + 2\sum_{n=1}^{N} \cos(nx). \tag{1}$$

We recognize this as the Dirichlet kernel, which we have seen can be written

$$g_N'(x) = \frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}.$$

From Figure 2.8 it seems that the error g_N is maximal at its smallest positive critical point, that is, at $x_N = \pi/(N+\frac{1}{2})$. If we can show that the error at this point remains positive in the limit $N \to \infty$, that is,

$$\lim_{N\to\infty} g_N(x_N) > 0, \tag{2}$$

then we can conclude that the Gibbs phenomenon holds for the function s.

We prove (2) using Riemann sums. Namely, we can write

$$g_N(x_N) = 2\sum_{n=1}^{N} \frac{\sin(nx_N)}{n} - (\pi - x_N) = 2\sum_{n=1}^{N} \frac{\sin(\xi_n)}{\xi_n}\,\Delta x - (\pi - x_N),$$

where $\xi_n = nx_N = n\pi/(N+1/2)$ are points at distance $\Delta x = x_N = \pi/(N+1/2)$ in the interval $0 < x < \pi$. By known facts on Riemann sums,

$$\lim_{N\to\infty} g_N(x_N) = 2\int_0^\pi \frac{\sin x}{x}\,dx - \pi.$$

This is approximately 0.562, in particular it is positive. A different proof, using (1), is indicated in Folland, Exercise 2.6.1.

We now know that the Gibbs phenomenon holds for the function s. This can be used to prove it for *any* piecewise C^1 but discontinuous f. We sketch how this can be done. Note first that $\frac{h}{2\pi}\,s(x-a)$ has a jump of height h at $x = a$. Suppose that the jumps of f in $0 \le x < 2\pi$ are $h_1, \dots, h_n$ at the points $a_1, \dots, a_n$. Then,

$$g(x) = f(x) - \sum_{j=1}^{n} \frac{h_j}{2\pi} s(x - a_j) \tag{3}$$

is piecewise C^1 and continuous, so its Fourier series converges uniformly on $\mathbb{R}$ (Folland, Thm. 2.5). On the other hand, close to a jump point a_k, one can show that the kth term in the sum (3) exhibits the Gibbs phenomenon, whereas all the other terms do not (the first statement follows from what we did above, but the second statement needs a little work). The Gibbs phenomenon for the kth term must then be cancelled by a corresponding Gibbs phenomenon for f. Thus, f exhibits the Gibbs phenomenon at each jump point.

Periodic solutions of ODE

In applications, one is often interested in finding periodic solutions to a problem. For instance, consider the ODE

$$x'' + 2x' + 5x = u(t),$$

with $u(t)$ 2π-periodic. This models the position $x(t)$ of a body (of mass 1) attached to a spring (with stiffness 5) experiencing some friction (with coefficient 2) and a periodic force $u(t)$, in suitable units. Alternatively, consider an inductor (inductance 1), a resistor (resistance 2) and a capacitor (capacitance 5) connected in series with a source of periodic voltage $u(t)$. The charge in the capacitor is then described by $x(t)$. In both examples, one expects that $x(t)$ is essentially 2π-periodic for large t. More precisely, as we shall se, any solution is the sum of a periodic solution and a function that tends to 0 rapidly as t grows.

We are thus led to the problem

$$x'' + 2x' + 5x = u(t), \qquad x(t) \quad 2\pi\text{-periodic.} \tag{4}$$

We will solve this in terms of Fourier series.

Suppose that u and x have Fourier series representations

$$u(t) \sim \sum_{n=-\infty}^{\infty} c_n e^{int},$$

$$x(t) \sim \sum_{n=-\infty}^{\infty} d_n e^{int}.$$

Suppose also that the series for x can be differentiated termwise, that is,

$$x'(t) \sim \sum_{n=-\infty}^{\infty} ind_n e^{int},$$

$$x''(t) \sim \sum_{n=-\infty}^{\infty} (-n^2) d_n e^{int}.$$

Plugging into (4) and identifying Fourier coefficients gives

$$(-n^2 + 2in + 5)d_n = c_n.$$

Thus, we are led to the formula

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{c_n}{-n^2 + 2in + 5} e^{int} = \sum_{n=-\infty}^{\infty} c_n G(in) e^{int}, \tag{5}$$

where

$$G(s) = \frac{1}{s^2 + 2s + 5}$$

is the Laplace transform of the impulse response/fundamental solution (see previous lecture notes).

This derivation is slightly formal. Let us indicate how to see that x really solves (4) when u is continuous and piecewise C^1. Then, (Folland, proof of Thm. 2.5) $\sum_{-\infty}^{\infty} |c_n| < \infty$. Using Weierstrass' test, it follows that both (5) and the series obtained by formally differentiating (5) once and twice converge uniformly. This guarantees that we are allowed to differentiate termwise twice in (5), so the derivation above is fine. However, even if u is less regular it still makes sense to consider (5) as the solution to (4). In particular, jumps in u will correspond to jumps in x''.

The same computation can be carried through for any equation

$$a_m x^{(m)} + a_{m-1} x^{(m-1)} + \cdots + a_0 x = u(t). \tag{6}$$

To be even more general, suppose that u has period $2\pi/\omega$, and seek solutions x with the same period. Writing

$$u(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t},$$

we find the periodic solution

$$x(t) = \sum_{n=-\infty}^{\infty} c_n G(in\omega) e^{in\omega t}, \tag{7}$$

where

$$G(s) = \frac{1}{a_m s^m + \cdots + a_0}.$$

Equivalently, writing $G(s) = |G(s)|e^{i\arg(G(s))}$ in polar form,

$$x(t) = \sum_{n=-\infty}^{\infty} c_n |G(in\omega)| e^{i(n\omega t + \arg(G(in\omega)))}.$$

This is useful to know for electrical engineers: signals with frequency $n\omega$ go through an amplitude modulation by $|G(in\omega)|$ and a phase shift by $\arg(G(in\omega))$. If the Fourier series is written in real form

$$u(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \big(A_n \cos(n\omega t) + B_n \sin(n\omega t)\big),$$

then the same principle applies:

$$x(t) = \frac{A_0 G(0)}{2} + \sum_{n=1}^{\infty} |G(in\omega)| \big(A_n \cos(n\omega t + \arg(G(in\omega))) + B_n \sin(n\omega t + \arg(G(in\omega)))\big).$$

Remark: Equation (4) breaks down if, for some n, $in\omega$ is a zero of the characteristic polynomial $a_m s^m + \cdots + a_0$. This is not a problem with our method but corresponds to an obstruction for the existence of periodic solutions to (6). In fact, you might remember from previous studies that solutions of, say,

$$a_m x^{(m)} + a_{m-1} x^{(m-1)} + \cdots + a_0 x = \cos(n\omega t)$$

will in this case typically contain terms like $t\cos(n\omega t)$ and $t\sin(n\omega t)$, and can thus never be periodic. Physically, this is an example of *resonance.*

In applications, one is particularly interested in the *stable* case, which means that all zeroes of the characteristic polynomial have negative real part. Then, all solutions of the homogeneous problem

$$a_m x^{(m)} + a_{m-1} x^{(m-1)} + \cdots + a_0 x = 0$$

tend to zero as $t \to \infty$ (a zero $a + ib$ leads to terms $e^{-at}\left\{\begin{smallmatrix}\cos(bt)\\ \sin(bt)\end{smallmatrix}\right\} p(t)$ in the homogeneous solution, with p a polynomial). In this situation, let x be an arbitrary solution of (6) and let x_{per} be the periodic solution (7). Then, $x - x_{\text{per}}$ will solve the homogeneous problem and will thus quickly tend to 0. So, regardless of initial data, *any* solution will approach the periodic solution x_{per}. In applications, x_{per} is called the *steady-state solution* and $x - x_{\text{per}}$ a *transient solution.*

Example: Let us return to the equation (4), with

$$u(t) = \begin{cases} 1, & 0 < t < \pi, \\ 0, & \pi < t < 2\pi. \end{cases}$$

In the example with the electric circuit, this corresponds to a constant voltage source, which is periodically turned on and off. The Fourier series of u is

$$\frac{1}{2}+\frac{2}{\pi}\sum_{k=0}^{\infty}\frac{\sin((2k+1)t)}{2k+1}.$$

The periodic solution is then

$$x_{\text{per}}(t)=\frac{G(0)}{2}+\frac{2}{\pi}\sum_{k=0}^{\infty}\frac{|G(i(2k+1))|}{2k+1}\sin\big((2k+1)t+\arg(G(i(2k+1)))\big),$$

where

$$G(s)=\frac{1}{s^2+2s+5}=\frac{1}{(s+1+2i)(s+1-2i)}.$$

Note that the zeroes of the denominator have negative real part, so any solution will quickly approach the periodic solution. We compute

$$\begin{aligned}|G(i(2k+1))| &= \frac{1}{|(1+i(2k+3))(1+i(2k-1))|}=\frac{1}{\sqrt{(1+(2k+3)^2)(1+(2k-1)^2)}}\\ &=\frac{1}{\sqrt{(4k^2+12k+10)(4k^2-4k+2)}},\end{aligned}$$

$$\arg(G(i(2k+1)))=-\arg(1+i(2k+3))-\arg(1+i(2k-1))=-\arctan(2k+3)-\arctan(2k-1).$$

This gives

$$x_{\text{per}}(t)=\frac{1}{10}+\frac{2}{\pi}\sum_{k=0}^{\infty}\frac{\sin\big((2k+1)t-\arctan(2k+3)-\arctan(2k-1)\big)}{(2k+1)\sqrt{(4k^2+12k+10)(4k^2-4k+2)}}.$$

Let us now compute the solution in closed form and compare. The solutions to $x''+2x'+5x=0$ are $e^{-t}(C\cos(2t)+D\sin(2t))$, while the solutions to $x''+2x'+5x=1$ are $\frac{1}{5}+e^{-t}(C\cos(2t)+D\sin(2t))$. Thus, for some constants,

$$x_{\text{per}}(t)=\begin{cases}\frac{1}{5}+e^{-t}(A\cos(2t)+B\sin(2t)), & 0<t<\pi,\\ e^{-t}(C\cos(2t)+D\sin(2t)), & \pi<t<2\pi.\end{cases}$$

The constants must be chosen so that x and x' are continuous at $t=\pi$ and so that the right limits of x and x' as $t\to 0$ agree with the left limits as $t\to 2\pi$. This gives four equations for the four unknowns A, B, C, D, and we find the solution

$$x_{\text{per}}(t)=\begin{cases}\dfrac{1}{5}-\dfrac{e^{-t}(2\cos(2t)+\sin(2t))}{10(1+e^{-\pi})}, & 0<t<\pi,\\[2ex] \dfrac{e^{\pi-t}(2\cos(2t)+\sin(2t))}{10(1+e^{-\pi})}, & \pi<t<2\pi,\end{cases}$$

which should be extended to a 2π-periodic function. Below we have plotted the exact solution in red together with the partial sum

$$\frac{1}{10}+\frac{2}{\pi}\sum_{k=0}^{3}\frac{\sin\big((2k+1)t-\arctan(2k+3)-\arctan(2k-1)\big)}{(2k+1)\sqrt{(4k^2+12k+10)(4k^2-4k+2)}}$$

in green. Even with so few terms in the Fourier approximation, the two curves are almost indistinguishable.