Exam MMG 710/TMA 362 (Fourier A.) 0–1

<u>Solution</u> 2016-08

1. (a) $F(\theta)$ is 2π-periodic since

$$\int_0^{2\pi} f(\varphi)\,d\varphi = \int_0^{2\pi} |\sin\theta|\,d\theta - \frac{2}{\pi}\cdot 2\pi$$

$$= 2\int_0^{\pi} \sin\theta\,d\theta - 4 = 2\cdot 2 - 4 = 0.$$

$F'(\theta) = f(\theta) = |\sin\theta| - \frac{2}{\pi}$ is even

$\Rightarrow$ $F(\theta)$ is odd up to a constant fct.

i.e. $$F(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin n\theta$$

$$f(\theta) = F'(\theta) = -\frac{4}{\pi}\sum_1^{\infty} \frac{1}{4n^2-1}\cos n\theta$$

$$\Rightarrow \quad b_n = -\frac{4}{\pi}\,\frac{1}{n(4n^2-1)}.$$

Now $F(\theta) = \int_0^{\theta} f(\varphi)\,d\varphi$, f is even, thus

$F(\theta) = -F(\theta)$, so $F(\theta)$ is indeed an odd function and $a_0 = 0$.

[Generally if $F(\theta) = \int_{\varphi_0}^{\theta} f(\varphi)\,d\varphi$ and $\varphi_0 \neq 0$, one has to integrate $F(\theta)$ to find a_0].

Answer $$F(\theta) = -\frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{n(4n^2-1)}\sin n\theta.$$

(b) We apply Parseval theorem for

$f'(\theta)$: $$f'(\theta) = -\frac{4}{\pi}\sum_1^{\infty} \frac{(n)}{4n^2-1}\sin n\theta,$$

$$\int_{-\pi}^{\pi} f'(\theta)^2\,d\theta = \left(\frac{4}{\pi}\right)^2 \sum_1^{\infty} \frac{n^2}{(4n^2-1)^2}\cdot \pi$$

$$f'(\theta) = \operatorname{sgn}(\sin\theta)\cos\theta$$

$$\int_{-\pi}^{\pi} f'(\theta)^2\,d\theta = \int_{-\pi}^{\pi} \cos^2\theta\,d\theta = \pi$$

Answer:

$$\sum_{1}^{\infty} \frac{n^2}{(4n^2-1)^2} = \left(\frac{\pi}{4}\right)^2 .$$

2. $b(x) = \sin^2 x = \frac{1}{2}(1 - \cos 2x)$

is a linear combination of

1 & $\cos 2x$, $a(x) = 1$

$\Rightarrow$ $\text{Span}\{a, b\} = \text{Span}\{1, \cos 2x\}$

& 1, $\cos 2x$ are orthogonal in

$L^2(-\pi, \pi)$.

$$f(x) = \cos^4 x = \frac{1}{2^2}(1 + \cos 2x)^2$$

$$= \frac{1}{2^2}\left(1 + 2\cos 2x + \frac{1}{2}(1 + \cos 4x)\right)$$

$$= \frac{1}{2^3}(3 + 4\cos 2x + \cos 4x)$$

with $\cos 4x \perp V = \text{Span}\{1, \cos 2x\}$

$\Rightarrow$ Projection of f on V is

$$\frac{1}{2^3}(3 + 4\cos 2x)$$

and the distance from f to V is

$$\frac{1}{2^3}\|\cos(4\cdot)\| = \frac{1}{2^3}\sqrt{\pi} .$$

3. Consider $\frac{1}{x^2+1}$ and its F. T.

$$\mathcal{F}: \quad \frac{1}{x^2+1} \longrightarrow \pi e^{-|\xi|}$$

$$\frac{e^{\pm ix}}{x^2+1} \longrightarrow \pi e^{-|\xi \mp 1|}$$

$$\Rightarrow \quad \frac{\cos x}{x^2+1} \longrightarrow \frac{\pi}{2}\left(e^{-|\xi-1|} + e^{-|\xi+1|}\right)$$

$$\int_0^\infty f(x)\,dx = [f(x) \text{ is even}] = \frac{1}{2}\int_{-\infty}^{\infty} f(x)\,dx$$

$$= [\text{def of F.T}] = \frac{1}{2}\hat{f}(0) = \frac{\pi}{2}(e^{-1}\cdot 2)$$

$$= \pi e^{-1}.$$

4. $u_0(t) = e^t$ is a solution to the inhomog. equation along with the homog. boundary value

$$u_0'(0, t) = u_0'(\pi, t) = 0 \quad \text{since}$$

u_0 is independent of x.

Write $u = u_0 + v$. Then v satisfies

$$\begin{cases} v_t = k v_{xx} \\ v_x(0,t) = v_x(\pi, t) = 0 \\ v(x, 0) = \cos^3 x - 1 \end{cases}$$

The general solution is of the form

$$v(x,t) = \sum_{n=0}^{\infty} e^{-n^2 t} a_n \cos(nx)$$

$$\cos^3 x - 1 = v(x, 0) = \sum_0^\infty a_n \cos(nx)$$

$$\text{LHS} = \cos x \,\frac{1}{2}(1 + \cos 2x) - 1$$

$$= \frac{1}{2}\cos 2x + \frac{1}{4}(\cos 3x + \cos x) - 1.$$

$$\Rightarrow a_0 = -1, \quad a_1 = \frac{1}{4}, \quad a_2 = \frac{1}{2}$$

$$a_3 = \frac{1}{4}. \quad a_n = 0, \quad n > 3.$$

Answer:

$$u(x,t) = e^t + \sum_{n=0}^{3} a_n e^{-n^2 t} \cos nx,$$

a_0, a_1, a_2, a_3 given as above

(5)

5. Perform partial fraction for $F(z)$:

$$F(z) = \frac{1}{z^2-5z-6} = \frac{1}{(z-6)(z+1)}$$

$$= \frac{1}{7}\left(\frac{1}{z-6} - \frac{1}{z+1}\right)$$

$\mathcal{L}^{-1}$: $\frac{1}{z-c} \to e^{ct}$

$$\frac{1}{z-6} \to e^{6t}, \quad \frac{1}{z+1} \to e^{-t}$$

$\mathcal{L}^{-1}$: $F(z) \to \frac{1}{7}(e^{6t} - e^{-t})$.

$\mathcal{L}: f * f \to F(z)^2 = \frac{1}{7^2}\left(\frac{1}{z-6} - \frac{1}{z+1}\right)^2$

$$= \frac{1}{7^2}\left(\frac{1}{(z-6)^2} + \frac{1}{(z+1)^2} - 2F(z)\right)$$

$$\Rightarrow f*f(t) = \frac{1}{7^2}\left(te^{6t} + te^{-t} - 2\frac{1}{7}(e^{6t} - e^{-t})\right)$$

(6)

$f*f(t)$

$$= \frac{1}{7^3}\left(e^{6t}(7t-2) + e^{-t}(7t+2)\right).$$

6. See the text book.