Exam in MMG710/TMA362. Fourier Analysis

2015-01-02. Solutions

1 (a) $f(x) = \pi - 2|x|, \quad -\pi < x < \pi.$

$$\Rightarrow \quad f'(x) = -2\,\mathrm{sgn}\, x = \begin{cases} -2, & x > 0 \\ 2, & x < 0 \end{cases} \quad \text{and}$$

$f(x)$ is not differentiable at $x = 0$.

$f(x)$ is continuous and piece-wise differentiable with $f'(x)$ piece-wise continuous. Thus the Fourier series of $f'(x)$ can be obtained by term-wise differentiation of that of $f(x)$:

$$f'(x) = -\frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}.$$

Now $f''(x) \equiv 0$ as a function in L^2. Thus the F-expansion of $f''(x)$ is $f''(x) \equiv 0$.

(b) The F-series of $f'(x)$ is not uniformly convergent since the sum $f'(x)$ is not continuous.

2. The best approximation of $f(x) = e^{-2x}$ by the functions $\{ae^{-x} + bxe^{-x}\}$ in $L^2(0\ \infty)$ is obtained by the orthogonal projection on the two dimensional subspace.

Write $u(x) = e^{-x}$. Let $v(x) = xe^{-x} + ae^{-x}$.

Seek a so that $u \perp v$, i.e $\langle u, v\rangle = 0$

$$0 = \langle u\ v\rangle = \int_0^\infty e^{-x}(xe^{-x} + ae^{-x})\,dx$$

$$= \int_0^\infty xe^{-2x}\,dx + a\int_0^\infty e^{-2x}\,dx$$

$$= 2^{-2}\int_0^\infty (2x)e^{-2x}\,d(2x) + a\tfrac{1}{2}$$

$$= 2^{-2}\int_0^\infty te^{-t}\,dt + a\tfrac{1}{2} = 2^{-2}\Gamma(2) + 2^{-1}a$$

$$= 2^{-2} + 2^{-1}a. \qquad a = -\tfrac{1}{2}.$$

The orthogonal projection of $f(x)$ is given by

$$\frac{\langle f\ u\rangle}{\langle u, u\rangle}u + \frac{\langle f\ v\rangle}{\langle v\ v\rangle}v.$$

$$\langle f, u\rangle = \int_0^\infty e^{-2x}e^{-x}\,dx = 3^{-1}, \quad \langle u\ u\rangle = \tfrac{1}{2},$$

$$\langle f\ v\rangle = \int_0^\infty e^{-2x}\left(xe^{-x} - \tfrac{1}{2}e^{-x}\right)dx = 3^{-2}\Gamma(2) - \tfrac{1}{2}3^{-1}$$

$$= 3^{-1}\left(3^{-1} - \tfrac{1}{2}\right) = -\tfrac{1}{3}\cdot\tfrac{1}{6} = -\frac{1}{3^2\cdot 2},$$

$$\langle v\ v\rangle = \int_0^\infty x^2 e^{-2x} - x e^{-2x} + \frac{1}{4} e^{-2x}\, dx$$

$$= 2^{-3}\Gamma(3) - 2^{-2}\Gamma(2) + \frac{1}{4}\cdot 2^{-1}$$

$$= 2^{-3}\, 2 - 2^{-2}\cdot 1 + \frac{1}{4} 2^{-1} = \frac{1}{8}$$

Answer: $\frac{2}{3} u + \left(-\frac{4}{3^2}\right) v = \frac{2}{3} e^{-x} - \frac{4}{3^2}\left(x e^{-x} - \frac{1}{2} e^{-x}\right)$

$$= \frac{4}{3} e^{-x} - \frac{4}{3^2} x e^{-x}.$$

3. (a) The Fourier transform of $\frac{\sin x}{x}$ is $\pi \chi_1(\xi)$.

Use Plancherel formula:

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\pi \chi_1(\xi)\right)^2 d\xi$$

$$= \frac{\pi}{2} \int_{-\infty}^{\infty} \chi_1(\xi)^2\, d\xi = \frac{\pi}{2} \int_{-\infty}^{\infty} \Pi_1(\xi)\, d\xi = \frac{\pi}{2} \int_{-1}^{1} 1\, d\xi = \pi.$$

(b) $\mathcal{F}: \quad \frac{\sin x}{x} \longrightarrow \pi \chi_1(\xi)$

$\Rightarrow \mathcal{F}: \quad \left(\frac{\sin x}{x}\right)^2 \longrightarrow (2\pi)^{-1}\left(\pi \chi_1 * \pi \chi_1\right)(\xi).$

Namely $\hat{f}(\xi) = (2\pi)^{-1} \pi^2 \int_{-\infty}^{\infty} \chi_1(\eta) \chi_1(\xi - \eta)\, d\eta$

$$= \frac{\pi}{2} \int_{-1}^{1} \chi_1(\xi - \eta)\, d\eta.$$

If $|\xi| > 2$ then $|\xi - \eta| \geq |\xi| - |\eta| > 2 - |\eta| > 1$ for any $\eta \in [-1\ 1]$, and thus $\chi_1(\xi - \eta) = 0$ for $\eta \in [-1, 1]$

$\Rightarrow \hat{f}(\xi) = \frac{\pi}{2} \int_{-1}^{1} 0\, d\eta = 0$.

4. We find first a special solution to the inhomogeneous equation with the homogeneous boundary conditions.

Ansats $\quad V(x) = \frac{1}{k}\sin x$. Then

$$k V_{xx}(x) + \sin x = k\,\frac{1}{k}(-\sin x) + \sin x = 0$$

and $V_t = 0$. $\quad V(0) = V(\pi) = 0$.

We see the general solution of the form

$$u(x\ t) = V(x) + v(x, t).$$

$v(x\ t)$ satisfies now the homogeneous equation & homogeneous boundary conditions along with the initial value

$$v(x\ 0) = u(x, 0) - V(x) = f(x) - \frac{1}{k}\sin x$$

$$= \sum_{n=1}^{\infty} b_n \sin nx = \sum_{n=1}^{\infty} c_n \sin nx, \quad c_n = b_n,$$

$$c_1 = b_1 - \frac{1}{k}.$$

$$v(x, t) = \sum_{k=1}^{\infty} e^{-kt} c_n \sin nx.$$

<u>Answer</u> $\quad u(x\ t) = \sum_{n=1}^{\infty} e^{-nt} c_n \sin nx + \frac{1}{k}\sin x$

$$c_n = b_n,\ n \neq 1, \quad c_1 = b_1 - \frac{1}{k}.$$

5. (a) We integrate the Fourier Series:

$f(x)$ is an even periodic function, its integration is a part from a constant function, an odd function, and we take $x > 0$ and integrate

$$\int_0^x (\pi - 2|t|)\, dt = \int_0^x (\pi - 2t)\, dt = \pi x - x^2 .$$

Thus
$$\pi x - x|x| + C = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^3} .$$

Evaluate at $x = 0$: $\quad C = 0$. Namely

$$\pi x - x|x| = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^3} .$$

Let $x = \frac{\pi}{2}$, $\sin(2n-1)\frac{\pi}{2} = (-1)^{n-1}$:

$$\pi^2\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^3}, \Rightarrow \sum_1^{\infty} \frac{(-1)^n}{(2n-1)^3} = \frac{-\pi^3}{2^5} .$$

To evaluate the second sum we use Parseval's theorem:

$$\left(\frac{8}{\pi}\right)^2 \left(\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}\right) \pi = \int_{-\pi}^{\pi} f(x)^2\, dx = 2\int_0^{\pi} (\pi - 2x)^2\, dx = \frac{2}{3}\pi^3 .$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^4}{3 \cdot 2^5} .$$