Fourier analysis, Fall 2015. Check-list for theory.

The complete course literature is as follows. [F] refers to Folland's book.

- (Motivations via Differential Equations.)

  [F]: Chapter 1.

- (Fourier series.) [F]: Chapter 2. 2.1-2.5. (Do *not* memorize Table 1; you may skip Thm. 2.6),
- (Hilbert space L^2-formulations) [F]: Chapt. 3.1–3.5 (skip the "dominated convergence theorem").
- (Applications to Boundary value problems) [F]: Chapt. 4.1–4.4.
- (Fourier Transform and applications) [F]: Chapt. 7.1–7.3. (7.1: only Thm. 7.1 is required; 7.2: only (7.16), skip the first version (7.15) of the inversion theorem on p. 218; 7.3: only pp. 229–231).
- (Laplace transform and applications) [F]: Chapt. 8.1–8.3.

Check-list of your progress and mastering of the course.

- What is a piecewise continuous function? What is a piecewise smooth function? (Folland's piecewise smooth is also called piecewise C^1.)
- Formulate the most important boundary value problems for the one-dimensional heat and wave equations. Explain how separation of variables leads to the problem of expanding the initial value as a Fourier series.
- What is the Fourier series of a periodic function? How does one pass between the complex and real form of a Fourier series? What is the Fourier cosine and Fourier sine series for functions defined on an interval $[0, T]$? What is the sum of these series *outside* the interval?
- Formulate and prove the Riemann–Lebesgue lemma for Fourier series.
- Formulate and prove the inversion formula for Fourier series ([F], Thm. 2.1).
- Formulate and prove a statement on differentiation of Fourier series ([F], Thm. 2.2 and 2.3).
- Formulate and prove a statement on integration of Fourier series ([F], Thm. 2.4).
- Formulate and prove a statement on uniform convergence of Fourier series ([F], Thm. 2.5).
- What is a complex inner product space? What is a Hilbert space? What, at least roughly, do we mean by $\ell^2(\mathbb{Z})$, $L^2([a, b])$ and $L^2(\mathbb{R})$?

- What is meant by convergence in norm, for "abstract" inner product spaces and in L^2?
- What is the relation between the following three: Convergence in $L^2([a,b])$, uniform convergence on $[a,b]$, pointwise convergence on $[a,b]$?
- Formulate and prove Bessel's inequality on general inner product spaces, formula (3.20).
- What is a complete orthogonal system? (cf. Thm 3.4)
- Which complete orthogonal system is related to complex Fourier series? Why is it complete (sketch of proof)? What does this mean (norm convergence, Parseval's formula)? What about Fourier sine and cosine series?
- What is the best way to approximate a vector in norm with linear combinations of vectors in a finite orthogonal system? Why? What does this tell us about Fourier coefficients?
- What is a self-adjoint operator? What can you say about eigenvalues and eigenvectors? (What Folland calls self-adjoint, I have called symmetric.)
- Give some criteria that guarantee that a second order ordinary differential operator, with boundary conditions, is self-adjoint.
- Give some criteria that guarantee that a second order ordinary differential operator, with boundary conditions, has a *complete* system of orthogonal eigenfunctions, in an appropriate L^2-space.
- What is convolution? Why is it commutative? Why is it associative?
- What is Fourier transform? Prove the rules for the Fourier transform of $f'(x)$, $xf(x)$, $f(x+a)$, $e^{iax}f(x)$ and for the Fourier transform of a convolution.
- Formulate the Riemann–Lebesgue lemma for Fourier transform. Sketch the proof, for Riemann integrable functions.
- Prove an inversion theorem for the Fourier transform ([F], Thm. 7.6).
- What is the Plancherel theorem for Fourier transform?
- What is Laplace transform?
- What is the Laplace transform of $f'(x)$ and $xf(x)$? What is the first and second shift rule? What is the Laplace transform of a convolution? Prove all these rules.
- Formulate and prove a uniqueness theorem for the Laplace transform.
- What is the fundamental solution (or impulse response)? How can the solution to an inhomogeneous equation be obtained from the fundamental solution?
- Derive Poisson's integral formula for a disc. (cf. Chapt. 4.4.)