MATEMATIK
Göteborgs universitet

js

Lösningar till tentamensskrivningen

MMG720, Differentialgeometri, 20190605

1. **Compute the curvature of the equiangular spiral, given in polar coordinates by the equation $r = ae^{b\varphi}$, where a and b are positive constants.**

We write $\boldsymbol{\gamma}(\varphi) = (a\cos\varphi e^{b\varphi}, a\sin\varphi e^{b\varphi})$ and compute

$$\begin{aligned}
\boldsymbol{\gamma}'(t) &= a(-\sin\varphi e^{b\varphi} + b\cos\varphi e^{b\varphi}, \cos\varphi e^{b\varphi} + b\sin\varphi e^{b\varphi})\ , \\
\boldsymbol{\gamma}''(t) &= a((b^2-1)\cos\varphi e^{b\varphi} - 2b\sin\varphi e^{b\varphi}, (b^2-1)\sin\varphi e^{b\varphi} + 2b\cos\varphi e^{b\varphi})\ .
\end{aligned}$$

Then $\|\boldsymbol{\gamma}'\| = a\sqrt{1+b^2}e^{b\varphi}$ and $\det(\boldsymbol{\gamma}', \boldsymbol{\gamma}'') = a^2(1+b^2)e^{2b\varphi}$. So

$$\kappa = \frac{1}{a\sqrt{1+b^2}e^{b\varphi}}\ .$$

Alternatively we use the formula for κ in polar coordinates. We have $r_\varphi = abe^{b\varphi}$, $r_{\varphi\varphi} = ab^2e^{b\varphi}$ so

$$\kappa = \frac{r^2 + 2r_\varphi^2 - rr_{\varphi\varphi}}{(r^2+r_\varphi^2)^{\frac{3}{2}}} = \frac{1}{a\sqrt{1+b^2}e^{b\varphi}}\ .$$

2. **Prove the Isoperimetric Inequality for simple closed curves; you may assume Wirtinger's Inequality.**

See the handout.

3. **Define the tangent space at a point P of a smooth surface S and show that it is a two-dimensional vector space.**

See Pressley, p. 85.

4. **Let $\boldsymbol{\gamma}\colon (\alpha,\beta) \to \mathbb{R}^3$ be a regular unit-speed curve with curvature $\kappa(s) \neq 0$ for all $s \in (\alpha,\beta)$. Let $\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ be its Frenet trihedron. The ruled surface**

$$\boldsymbol{\sigma}(s,v) = \boldsymbol{\gamma}(s) + v\boldsymbol{n}(s), \qquad (s,v) \in (\alpha,\beta)\times\mathbb{R}\ ,$$

is called the principal normal surface.
a) Determine the points where the parametrisation $\boldsymbol{\sigma}$ is not regular.
b) Compute the Gaussian curvature K. **(5p)**

We compute

$$\begin{aligned}
\boldsymbol{\sigma}_s &= \boldsymbol{\gamma}'(s) + v\boldsymbol{n}'(s) = \boldsymbol{t}(s) + v\tau\boldsymbol{b}(s) - v\kappa\boldsymbol{t}(s) \\
\boldsymbol{\sigma}_v &= \boldsymbol{n}(s) \\
\boldsymbol{\sigma}_{sv} &= \tau\boldsymbol{b}(s) - \kappa\boldsymbol{t}(s) \\
\boldsymbol{\sigma}_{vv} &= 0
\end{aligned}$$

This gives $\boldsymbol{\sigma}_s \times \boldsymbol{\sigma}_v = (1-v\kappa)\boldsymbol{b} - v\tau\boldsymbol{t}$. The parametrisation is not regular if $1 - v\kappa = v\tau = 0$, so $\tau = 0$ och $v = 1/\kappa$.
We have $E = (1-v\kappa)^2 + v^2\tau^2$, $F = 0$, $G = 1$, $EG - F^2 = (1-v\kappa)^2 + v^2\tau^2$ and $N = 0$, $M = \det(\boldsymbol{t} + v\tau\boldsymbol{b} - v\kappa\boldsymbol{t}, \boldsymbol{n}, \tau\boldsymbol{b} - \kappa\boldsymbol{t})/\sqrt{(1-v\kappa)^2 + v^2\tau^2} = \tau/\sqrt{(1-v\kappa)^2+v^2\tau^2}$. So $K = LN - M^2/EG - F^2 = -\tau^2/((1-v\kappa)^2 + v^2\tau^2)^2$.

5. a) Formulate the Gauss-Bonnet Theorem for curvilinear polygons.
b) Verify the Theorem for the closed curve in the plane consisting of a half circle with radius r and a diameter of the circle, by computing both sides of the formula separately.

a) Let $\boldsymbol{\gamma}$ be a positively-oriented unit-speed curvilinear polygon with n edges on a surface $\boldsymbol{\sigma}$, and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the interior angles at its vertices. Then

$$\int_0^{l(\boldsymbol{\gamma})} \kappa_g ds = \sum_{i=1}^{n} \alpha_i - (n-2)\pi - \int_{\mathrm{int}(\boldsymbol{\gamma})} K d\mathcal{A}_{\boldsymbol{\sigma}} \ .$$

b) For a circle in the plane the geodesic curvature is the curvature so $\kappa_g = 1/r$ and $\int_0^{l(\boldsymbol{\gamma})} \kappa_g ds = \int_0^{\pi r} \frac{ds}{r} = \pi$. As $n = 2$, the interior angles both equal to $\frac{\pi}{2}$ and $K = 0$, the right-hand side is equal to $2 \cdot \frac{\pi}{2} = \pi$.

6. Describe (qualitatively) the geodesics on on a torus.

See Pressley p. 443.