Analytic Geometry and Linear Algebra.
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Now the basic objects of geometry can be described in terms of coordinates.
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2. A line through the point $a \in \mathbb{R}^n$ with direction $v \in \mathbb{R}^n$ is a set $L_{a,v} := \{a + tv; t \in \mathbb{R}\}$. 

3. The distance between two points $x$ and $y$ is $|x - y| = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}$.

4. A circle or sphere is the set of points that satisfy $|x - c| = R$ for a fixed center $c$ and radius $R$.

5. The angle between two directions $v$ and $w$ is given by $\arccos\left(\frac{v \cdot w}{|v||w|}\right)$, where $v \cdot w = \sum v_j w_j$. 

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There is an extra (unexpected ?) bonus with the translation to coordinates: We can do geometry in any dimension, and it is in principle as easy as in two dimensions. Here is an example of this:
The method of least squares

Let \((x_1, y_1), \ldots (x_n, y_n)\) be a number of points in the plane. If \(n > 2\) we cannot draw a line through all of the points in general.
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Two numbers, \(a\) and \(b\) determine the line \(y = ax + b\). Instead of trying to solve the overdetermined system of equations

\[
y_j = ax_j + b
\]

we try to minimize the error

\[
\epsilon^2 = \sum_j (y_j - (ax_j + b))^2
\]

over all choices of \(a\) and \(b\).
Let \( x = (x_1, \ldots, x_n) \), \( 1 = (1, 1 \ldots 1) \) \text{(two points in } \mathbb{R}^n!! \text{)} and let \( P \) be the twodimensional plane

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$$ P = \{ a\mathbf{x} + b\mathbf{1}; a, b \in \mathbb{R} \}. $$

Any line in $\mathbb{R}^2$ corresponds to a choice of $a, b$ and therefore to a point in $P$. The minimal error $\epsilon$ that we want to find is the distance from the point $\mathbf{y}$ in $\mathbb{R}^n$ to the plane $P$. Why?

The distance from $\mathbf{y}$ to the plane is

$$ d = \min |\mathbf{y} - \mathbf{z}|, $$

where $\mathbf{z}$ ranges over all points in the plane $P$. But, any point $\mathbf{z}$ in the plane is of the form $\mathbf{z} = a\mathbf{x} + b\mathbf{1}$, so $d = \epsilon$. How do we find it?
Let $x = (x_1, \ldots, x_n)$, $1 = (1, 1, \ldots, 1)$ (two points in $\mathbb{R}^n$!!) and let $P$ be the twodimensional plane

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How do we find it?
It is clear from a figure that the minimum will occur in a point \((a_0, b_0)\) such that \(y - (a_0x + b_01)\) is perpendicular to any vector in the plane. (Exercise: prove this!).
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This is a homogenous system of two equations and two unknowns which always has a solution. Observe that \(a_0\) and \(b_0\) are the unknowns, and \(x, y\) are given!
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\[ ax^2 + bx + c \] (we now get three equations with three unknowns instead), or any other type of functions involving exponentials, trigonometric functions or ....
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Notice that it solves a problem in the plane by using geometry in $n$ dimensions, where $n$ is the number of points and can be arbitrary big. The method of least squares was probably first used by Gauss, who applied it to find a 'lost planet'.
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The basic problem that the least squares method addresses is to describe data with many degrees of freedom (the points \((x_i, y_i)\)) approximately with few parameters \((a \text{ and } b)\).
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A similar problem arises when we try to compress a picture with many pixels to few kilobytes. This is where ‘least sums’ have proved to be surprisingly useful.
One central topic in linear algebra is the solution of linear systems of equations

\[ a_{11} x_1 + \ldots a_{1n} x_n = y_1 \]
\[ a_{21} x_1 + \ldots a_{2n} x_n = y_2 \ldots \]
\[ a_{m1} x_1 + \ldots a_{mn} x_n = y_m \]

or

\[ Ax = y, \]

where \( A \) is the coefficient matrix of the system.
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\begin{align*}
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\]

or

\[Ax = y,\]

where \(A\) is the coefficient matrix of the system. Here is the most important theorem in that context. We think of \(A\) as a linear map \(x \mapsto Ax\) from \(\mathbb{R}^n\) to \(\mathbb{R}^m\). Recall that \(\text{Ker}(A) = \{x; Ax = 0\}\) and \(\text{Im}(A) = \{Ax; x \in \mathbb{R}^n\}\); they are both linear subspaces of \(\mathbb{R}^n\) and \(\mathbb{R}^m\) respectively.
Theorem

Let $A$ be a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$. Then

$$\dim(\text{Ker}(A)) + \dim(\text{Im}(A)) = n.$$
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The statement and the proof hinges on the notion of dimension. A linear space, like $\mathbb{R}^n$ has many different bases, but they have all the same number of elements. (Exercise: Prove this!) This is the dimension of the space. Say the dimension of $Ker(A)$ is $k$, and let $e_1, \ldots, e_k$ be a basis. We can find vectors in $\mathbb{R}^n$, $f_1, \ldots, f_{n-k}$ that complete $e_1, \ldots, e_k$ to a basis of $\mathbb{R}^n$. Let $F$ be the linear span of $f_1, \ldots, f_{n-k}$. Then the restriction of $A$ to $F$ is injective (why?).
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The theorem can be reformulated in the following way. Let \( G = \mathbb{R}^m/\text{Im}(A) := \text{coker}(A) \). Then

**Theorem**

\[
\text{ind}(A) := \dim(\text{Ker}(A)) - \dim(\text{cokernel}(A)) = n - m.
\]

The advantage with this formulation is that the kernel and the cokernel may have finite dimensions even if \( A \) acts on an infinite dimensional space.

If \( A : V \rightarrow V \) where \( V \) is a vector space of finite dimension, then the index is always zero. This is not always the case in infinite dimensions as we shall see later. The index is an important object to study in the theory of partial differential equations, when \( A \) is a differential operator.
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Matrices as we have seen arise in the study of linear maps between finite dimensional vector spaces, but they also appear in a somewhat different context.

Let $Q(x) = \sum a_{ij} x_i x_j$ be a quadratic form. If $A = (a_{ij})$ we may write $Q(x) = x^t Ax$, and we may assume that $A$ is symmetric.

If we change basis in $\mathbb{R}^n$, $x = My$, where $M$ is an invertible matrix, we have $Q(x) = y^t M^t A My = Q'(y)$.

We now have the second important theorem of linear algebra: Theorem We may find an (orthonormal) $M$ such that $Q'(y) = \sum \lambda_j y_j^2$. 

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We now have the second important theorem of linear algebra:

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*We may find an (orthonormal) \( M \) such that*

\[ Q'(y) = \sum \lambda_j y_j^2. \]
This is the *Spectral Theorem*. If we interpret $A$ as a linear operator, $A' = M^{-1}AM$ is the matrix for the same operator in the new basis, where $y$ are coordinates. But, since $M$ is orthonormal, $M^t = M^{-1}$. hence the theorem says that we change coordinates so that $A'$ is the diagonal with eigenvalues $\lambda_j$. 


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We are now ready to discuss the corresponding facts in infinite dimension.
Infinite dimension and Hilbert space.
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However, just being a linear space is not enough structure to give interesting or useful mathematics. The interest starts when we introduce geometry, i.e., have a way to measure distances.
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In other words, there is an element $u$ in the space such that

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**Example 1**: Let
\[ V = \{ u = (u_0, \ldots, u_n, \ldots), \quad u_k = 0 \quad \text{for } k \text{ sufficiently large} \} \]
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Example 2 is complete, Example 1 is not.
Every Hilbert space $V$ has an orthonormal basis, i.e., there is an orthonormal set of vectors $\{e_\alpha\}_{\alpha \in \Lambda}$ such that any vector in $V$ can be written

$$x = \sum_{\Lambda} c_\alpha e_\alpha,$$

and

$$\|x\|^2 = \sum_{\Lambda} |c_\alpha|^2.$$
Theorem

Every Hilbert space $V$ has an orthonormal basis, i.e., there is an orthonormal set of vectors $\{e_\alpha\}_{\alpha \in A}$ such that any vector in $V$ can be written

$$x = \sum_A c_\alpha e_\alpha,$$

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In practice, the most interesting case is when $A$ is countable. The Hilbert space is then said to be separable.
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Briefly, there is only one Hilbert space.
Corresponding to matrices we now have linear maps, or operators, $A : V \to V$. Let $B = \{x; \|x\| \leq 1\}$ be the unit ball in $V$. 

1. We say that $A$ is bounded if $A(B)$ is bounded.
2. We say that $A$ is compact if $A(B)$ is compact.
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Let $A$ be a compact operator on a Hilbert space $H$. Assume $A$ is selfadjoint, i.e.

$$(Ax, y) = (x, Ay).$$

Then the quadratic form $(Ax, x)$ can be diagonalized. This means that there is an orthonormal basis $(e_j)$ of eigenvectors of $A$. 
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Then the quadratic form $(Ax, x)$ can be diagonalized. This means that there is an orthonormal basis $(e_j)$ of eigenvectors of $A$. Moreover, the eigenvalues $\lambda_j$ tend to 0.
As our first example of a Hilbert space we take $L^2(T)$, the space of square integrable functions on the circle, with norm given by

$$\int_T |f|^2 d\theta = \|f\|^2.$$
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This corresponds to the linear map $Af = -f''$, which is not bounded and certainly not compact. Nevertheless the theorem applies, essentially because the inverse of $A$ is compact. Hence there is a basis of eigenvectors, namely $e_j(\theta) = e^{ij\theta}$.
The eigenvalues of $A$ are $\lambda_j = j^2$ and any function can be written

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This ‘explains’ Fourier analysis but has much wider scope. E.g. we can consider instead a domain $D$ in the plane and the Hilbert space of functions that are square integrable on $D$, with the quadratic form

$$\int_D |\nabla f|^2.$$
Let us continue with the last example and consider the plane in our Hilbert space

\[ P := \{ f; f = g \text{ on } \partial D \}, \]

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where \( g \) is given. Let \( f_0 \) be an element in \( P \) that minimises \( Q(f) \). Think of it as the element in \( P \) of smallest ’norm’, where ’norm’ squared is \( Q(f) \).
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Then \( f_0 \) must be ‘orthogonal’ with respect to \( Q \) to any vector \( u \) in the plane \( P - f_0 \). Such functions \( u \) are of the form \( u = f - f_0 \), where both \( f \) and \( f_0 \) equal \( g \) on the boundary of \( D \), i.e. they are just functions that vanish on the boundary of \( D \).
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Hence

\[ 0 = Q(f_0, u) = \langle (−\Delta f_0), u \rangle, \]

where \( \Delta = \partial^2 / \partial x^2 + \partial^2 / \partial y^2 \) is the Laplace operator.
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If this holds for all \( u \) that vanish on the boundary, \( \Delta f_0 = 0 \) (and \( f = g \) on the boundary). So, we have solved Dirichlet’s problem.
Notice how similar this is to the method of least squares.
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Building on work of *Ivar Fredholm* Hilbert also considered equations of the form

$$(\lambda I - T)f = g,$$

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Building on work of *Ivar Fredholm* Hilbert also considered equations of the form

$$ (\lambda I - T)f = g, $$

where $\lambda$ is a number and $T$ is a compact operator. A typical compact operator is

$$ Tf(x) = \int K(x, y)f(y)dy, $$

where $K$ is continuous. This is the integral version of an operator given by matrix multiplication.
Theorem

(The Fredholm alternative) Let

\[ Tf(x) = \int K(x, y)f(y)\,dy, \]

where \( K \) is continuous. Then, for any complex number \( \lambda \), either the equation

\[ (\lambda I - T)f = g \]

has a solution \( f \) for any choice of \( g \), or the equation

\[ (\lambda I - T)f = 0 \]

has a non trivial solution.
The second alternative means that $\lambda$ is an eigenvalue of $T$. 
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Fredholms article was published in 1903 and inspired Hilbert’s general theory on integral equations and the solvability of ‘equations in infinitely many variables’. (1912).
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Fredholm’s article was published in 1903 and inspired Hilbert’s general theory on integral equations and the solvability of ‘equations in infinitely many variables’. (1912).

The next big step was John von Neumann’s general theory of Hilbert spaces (he introduced that name) as a foundation of quantum mechanics in 1932 (when von Neumann was 29 years old).
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In this way we can see Hilbert space as the mathematical theory of quantum mechanics, similarly to how Riemannian geometry is the mathematics of the theory general relativity. We shall next turn to the mathematics of classical mechanics, i.e., calculus.