Third Lecture

Alexander the Great and the Hellenistic period

Traditionally Greek Society consisted of small city states, the most notable being Athens and Sparta, as well as various colonies along the Mediterranean and the Black Sea. Those states were not seldom in conflict with each other, conflicts that occasionally resulted in war, the most famous being the Peloponnesian War between Athens and Sparta and their respective allies. It started in 431 B.C. and lasted in stages for almost thrity years until 404 B.C. Sparta emerged as the victor but was not able to maintain its supremacy, and Athens revived. Next to Greece was the powerful Persian empire, very different in spirit and a constant threat. As usual a powerful external enemy provides an occasion for rallying to a common cause. The Persians attacked Greece repeatedly during the 5th century B.C. but were always ultimately expelled. And then, as out of nowhere, more specifically Macedon, a peripheral Greek state, whose inhabitans were semi-barbarians, meaning that they could not speak Greek properly, emerged under its ruler Philip II, and conquered a large part of Greece. Philip was murdered and his son Alexander, later to be known as the Great, assumed the throne at the age of twenty and completed his father's project of conquering the whole of Greece, and thereby unifying it. Alexander born in 356 B.C. had had as his teacher Aristotle in his early youth, thereby establishing a connection with the philosophers of Classical Greece, making him not only a military commander but also a proponent of Greek culture. His ambitions were thus not only confined to Greece. Two years later he crossed into Asia never to return. He defeated the Persians under Darius III overthrew him and subjugated the empire, furthermore he conquered Egypt and the lands of Tigris and Eufrat, proceeding east to India, where Elephants were encountered and would from then on play a conspicious role in calssical warfare. Most importantly Greece culture spread, tangibly manifested by the founding in 332 B.C. of the city of Alexandria in the Nile delta, known for its extensive library. After the death of Alexander, his wide ranging empire disintegrated, there being no heir. The main empires emerging were the Ptolemaic centered in Egypt, the Seleucid in the east, including central Asia, and the Pergamon and Macedonian in the West comprising the larger part of present day Turkey and Greece. The most important thing was that the center of gravity of Greek culture and science moved from Athens to Alexandria. Greece would never be the same again. The classical Greek period is known as the Hellenic, and the one following Alexander the Hellenistic. Greek culture would have a strong influence on Rome and live on through the Eastern Roman empire merging into the Byzantine, and Greek ethnic presence would prevail far into eastern Turkey until the beginning decades of the 20th century.

As noted the center of gravity became Alexandria, which was turned into a cosmopolitan city, involving a mixture not only of races and ethnic groups, but also a disolution of slaves and masters, being less class-conscious than classical

Greek society. While books were a rarity in Greece they become much more common, due to the papyrus, in Alexandria. The above mentioned library is rumored to have had about 750'000 volumes, and was engaged in a project of copying any books which came its way. The famous burning of the library is to a large extent apocryphal, true it was burned and destroyed at various times, but revived, and although it eventually fizzled out, there is no precise point in time on which to hang its demise. The city was in many ways quite modern with a lot of ingenious mechanical contraptions not only to easen everyday life but to provide entertainment and awe, such as moving parts in temples or vehicles powered by steam.

Classical Hellenistic mathematicians

Euclid no doubt got his mathematics from the Plato academy and although he settled and worked in Alexandria he is of the classical tradition. The same can also be said of Apollonius, although he too worked in Alexandria during the early Hellenistic period. He is known for his systematic treatment of the conic sections. Those did not originate by him, but had a long history, Euclid among others write on them as well, but his presentation of them is masterly and from a deductive point impeccable. He does not use algebra, although the modern way of using equations is implicit, but reasons solely synthetically, requiring great ingenuity at each step. He incidentally was the one who introduced the terminology of 'ellipse', 'parabola' and 'hyperbola' noting as the first that the latter should be considered by both its branches. As ought to be well-known they refer to 'less than', 'equal' and 'more than', a triparte division forming a common theme through much of mathematical classification ever since. An elliptic expression is a shortened form, while hyperbole is exaggeration. A parable, as common in the New Testament, provides an 'equality' between two phenomena, more precise than a metaphor or an analogy. Conic sections do not naturally occur by taking say wooden cones and producing plane sections by a saw, but when shadows or lights fall on oblique walls. It provides an entirely new way of creating curves not by ruler and compass, although the ellipse can of course be created mechanically by a fixed string nailed at two points, the foci. It is planar geometry but created in a three-dimensional setting. Supposedly Apollonius did not explore the topic because of its intrinsic beauty, although this is what emerges from his work, but for the applications of solving geometric problems such as the trisection of an angle or the duplication of a cube, all of which can be effected by intersections of conics,thus greatly extending the scope of constructions by ruler and compass, which only solves quadratic equations if repeatedly.

Hellenistic mathematicians

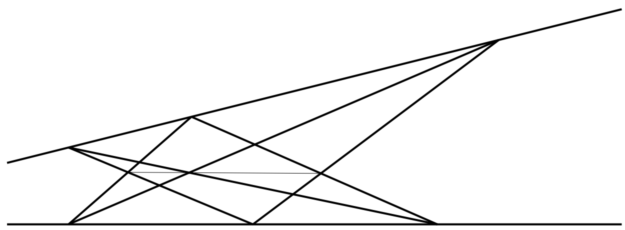
The post-classical mathematicians of the Hellenistic period had a different attitude to mathematics than the classical ones. For one thing they were less of purists, more concerned with applications, not staying away from numerical calculations. Deductive reasoning was not abandoned, but there was also more of an indulgence towards heuristic reasoning, as there was an impatience to get results, because the scope of Hellenistic geometry was larger than the Classical. Yet heuristic reasoning, being a throw back to the old Egyptians occasionally gives wrong formulas.

One example of a Hellenistic mathematician was Heron, known mainly for his formular of the area of a triangle in terms of its sides. If s is half the perimeter and the sides are given by a, b and c the formula is

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

By the congruence theorem that three sides determine a triangle, a formula should exist. The area of the triangle will be zero when $s = 0$ or more generally $s = a, b$ or c. From this it is natural to set up $s(s-a)(s-b)(s-c)$ but this scales as the fourth power, instead of the second power, and thus it is natural to take the square root. This is hardly deductive reasoning, at most a logically generous example of heuristic. How should one think of it? A mnemonic principle? This was not the way Heron came up with the formula.

Another example is Pappus, who among other things is known for the following theorem. Given three pairs of points on two distinct lines in the plane, the three intersection points, also lie on a line.



A kind of result which would not appear in classical Euclidean geometry and foreshadows projective geometry which would come to the fore later.

Another result contributed to him is that the volume of generated by the revolution of a closed curve along an axis that does not intersect it is given by the area of the enclosed curve times the circumference of the circle traced by its center of gravity. No known proof of Pappus exists and the result may very well have been known before him.

Archimedes

Archimedes was undeniably one of the most astounding humans of antiquity. Not only was he one of the foremost mathematicians to appear on the human scene, he was also so much more. An engineer and inventor, whose inventions serving military needs (if not of conquest and domination but of defense) became legendary. More peaceful inventions such as the screw to pump up water are still used today. He was the first to put down principles of mechanics, especially those pertaining to the lever (famous are his words: 'Give me a fixed point and I will move the Earth'), and hydrodynamics, especially what is now known as the law of Archimedes (the story of his solution, running naked in the city, shouting 'Eureka', are known to most of us.) What is noteworthy is that in his presentation of those results he tried to follow the deductive style of Euclid, by first presenting some general simple principles, out of which the final results are derived by deductive thinking. As with all such ambitions the result is a transparent presentation which can be critizied. It is also noteworthy that he found no need to draw a line between the science of space - geometry, and that of physics. Indeed what distinguishes mechanics from geometry. In the latter we are studying something static, in which time does not enter, in the former something dynamic, in which precise movement are essential. However, it is noteworthy that the intuition that is required is quite different in mechanics, as compared to geometry, in the latter, the visual one seems to be enough, while in mechanics some kind of muscular intuition apparently enters. The ambition to present physics in the style of Euclid would prevail, and Newton's Principia is written in that style. It bespeaks an intention to master the physical world through deductive thinking anchored in a few strategically chosen empirical facts, and to extract some simple principles. It is no doubt the feasibility of this, which can explain the spectacular success of the physical sciences, to which we will return later. Let us just observe that the ultimate synthesis of geometry with physics was achieved by the Relativity theory of Einstein.

Archimedes was born in Syracuse, a Greek settlement on Sicily in 287 B.C., he was educated in Alexandria but did return to Syracuse. That settlement was attacked by the Romans, during which he had the opportunity to show off the skills of a military engineer, to which we have already alluded. The Greek could not withstand the superiority of the Roman attackers, and supposedly Archimedes perished in the aftermath - 212 B.C. According to legend[1], he was attacked and killed by a Roman soldier, admonishing him not to disturb the circles he had traced in the dust. Whether the story is true or not, is of less importance, than what it adds to his legend. Mathematical contemplation is seen as something transcending quotidian affairs, even those engaging a victorious army. We can only hope that the distraught commander of the Roman troops, had this hot- and thick-headed soldier executed for his transgression. Archimedes was clearly worth his weight in gold, to anyone considering a serious military venture. It should, however, be stressed, that admired as those inventions were of the general public, Archimedes himself disdained them, as being mere diversions. Pure mathematics was his metier and great love.

Now striking as his more practical achievments may be, we should concentrate on his mathematical work. The method of exhaustion was already employed by Euclid, and hence there is good reason to suspect that it was known before him. In the hand of a master, such as Archimedes, it was employed to perfection. However, each use required great ingenuity, and Archimedes did not develop a more systematic tool, as did Newton and Leibniz nearly two-thousand years later, which would allow lesser men and women to routinely achieve results, which would have baffled an Archimedes. There has been no lack of speculations as to how close Archimedes actually was to achieving such a theory, and also some daring suggestions to the effect that had such a method been made available, the course of human development would have been advanced by two millenia. Personally I find such suggestions somewhat naive and simplistic, but anyone is entitled to their opinions on the matter. It certainly puts mathematics and mathematical progress in an exalted position.

Remarkable as the results of Archimedes may have been, given the time at which he achieved them, what is more remarkable and enduring is the ingenuity he displayed, invoking not only geometrical arguments but also dynamical, mechanical ones, giving witness to a fertile imagination ranging freely over disciplines.

The most basic one concerns the comparison of areas and volumes of a sphere and that of a circumscribed cylinder, which so excited him that he reportedly wished to have the accompanying figure on his tomb[2] This result is now easily proved by any first year student of calculus, at the time maybe only an Archimedes might have been able to find it.

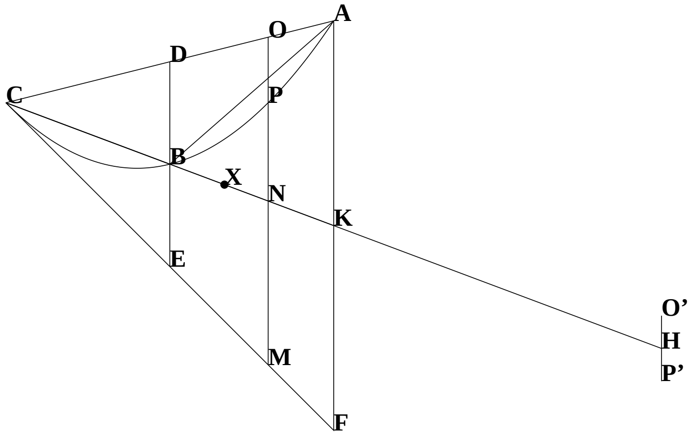
The works of Archimedes

To get an idea of the range and productivity of this great sage, let us include a table of contents to his works.

- Arithmetics *Measurement of a Circle, Sandrecogner*
- Mechanical construction of curves *The problem of* $\nu\epsilon\nu\sigma\iota\zeta$
- Cubic Equations (by means of intersection of conics)
- Volumes and areas of various geometric figures (parts of spheres, parabolids, spirals
- On the Sphere and Cylinder
- Measurement of a circle ($3\frac{10}{71} < \pi < 3\frac{1}{7}$)
- Conoids and Spheroids
- On spirals
- On the Equilibrium of planes
- The Sand-reckoner
- Quadrature of the Parabola
- On Floating Bodies
- Book of Lemmas
- The Method

As an illustration of his style we will consider in some detail his derivation of the area of a parabolic segment.

Consider the figure below.

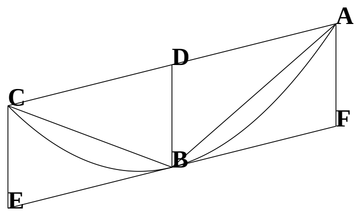


We have a parabolic arc ABC where B is chosen such that the area of the triangle is maximal. This is easily seen to be the case if the tangent at B is parallel to the linear segment AC. Now form the triangle ACF by taking the tanget at C and the diameter through A. (By the diameter of a parabola is meant any line parallel to its axis[3]). Now form the line through C and B which will intersetct AF at K. Extend the line to H such that the length of CK is equal to KH, and think of it as a lever with its fulcrum at K. Now for any point P on the parabolic arc, think of the linesegment OP made up of the diameter through P, and move it to O', P'. Now it is a property of the parabola that we have the proportions $KH : KN$ and $OM : OP$ are equal (or if you prefer $(HK)(OP) = (KN)(OM)$). This can be interpreted as that the weight of the segment $O'P'$ exactly balances the weight of OM placed at N. Doing it for all the points P on the arc, we are in effect moving the parabolic segment as a weight to H balancing the entire triangle ACF. Now the latter can be replaced by having all its weight moved to the center of gravity X on the line. (We need to observe that the line CK is a median, i.e. $KF = KA$). As X is placed one third along the line KC counted from K we conclude that the weight, i.e. the area of the triangle ACF is three times that of the parabolic segment. Now the area of the triangle ABC is half of that of ACK (they have the same base AC while the height of one is half of the other), while that of ACK is obviously half of ACF. Thus the area of the parabolic segment is 4/3 of the triangle.

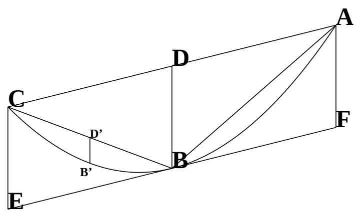
Now Archimedes did not think of this as a proof, only a so called heuristic method of getting to the proof. The remarkable thing is the mechanical imagination entering the picture. Why did he not think of it as a rigorous proof? Obviously he thought of those figures as made up of line segment. To get the area we only needed to add up all the lengths of the segments, which we could think of as weights as well, getting the weights of the areas. (As they are going to be compared it does not matter what exchange rate we use). But as the areas of the lines are zero, how could they add up? or if there is an infinite number of line segments what do we mean by adding them up? That the argument is fallacious is easy to see, given any two rectangles with the same height, but different bases, they would be shown to have the same area, as we can easily get a 1-1 correspondence between segments of equal height.

Could we make Archimedes argument rigorous?

Archimedes indicated another way.



Let AC be a chord of a parabola. It determines a parabolc segment, whose area we want to compute, or rather to compare with that of a simpler figure. For that purpose let DB be a diameter of the parabola bisecting all chords parallel to AC [4]. Then form a parallelogram $ACEF$ by letting BF and BE be the length of $DA(= DC)$. Clearly the triangle ACB has half the area A of the parallelogram. Now we can proceed to do the same construction on the chords BC and BA



The thing to note is that the length of the segment $B'D'$ is just one fourth of the length of BD, this being a characteristic property of the parabola. This means that the area of the triangle CBB' is one quarter of the triangle CDB. The same thing will of course be true on the other side. Hence the areas of the two triangles we add will be one fourth of the area of the original triangle CBA. Thus if we denote the latter area by Δ we are going to sum the infinite geometric sum

$$\Delta + \frac{1}{4}\Delta + \frac{1}{4^2}\Delta + \frac{1}{4^3}\Delta \ldots$$

which gives $\frac{1}{1-\frac{1}{4}}\Delta = \frac{4}{3}\Delta$. But Archimedes would not be so rash. He would note that in a geometric series with ratio $\frac{1}{4}$ we would have the identity

$$\Delta + \frac{1}{4}\Delta + \ldots + \frac{1}{4^n}\Delta + \frac{1}{3}\frac{1}{4^n}\Delta = \frac{4}{3}\Delta$$

[5] in other words an exact error term at the end being a third of the last term $\delta_n = \frac{1}{4^n}\Delta$. He would then argue that if the area A of the segment satisfy $A > (4/3)\Delta$ then we could find a finite sum of triangles whose area S would satisfy $A > S > \frac{4}{3}\Delta$, because the triangles exhaust the area of the segment. Being finite means that for some n we would have $S + \frac{1}{3}\delta_n = \frac{4}{3}\Delta$ hence $S < \frac{4}{3}\Delta$ which gives a contradiction. Similarly if $A < (4/3)\Delta$ we can look at the number

$\frac{4}{3}\Delta - A > 0$. As the numbers δ_n can be arbitrarily small[6] we can find n such that $\frac{4}{3}\Delta - A > \delta_n$. We can then find a finite sum of triangles such as $\Delta + \frac{1}{4}\Delta + \ldots + \frac{1}{3}\delta_n = \frac{4}{3}\Delta$ hence $A < \Delta + \ldots \delta_n$ which is absurd as it would imply that the segment is contained in a finite number of triangles. Of course what Archimedes does is to derive the limit of an infinite geometric series in a very careful way, making precise by the geometric problem what is supposed to be meant by such an infinite sum.

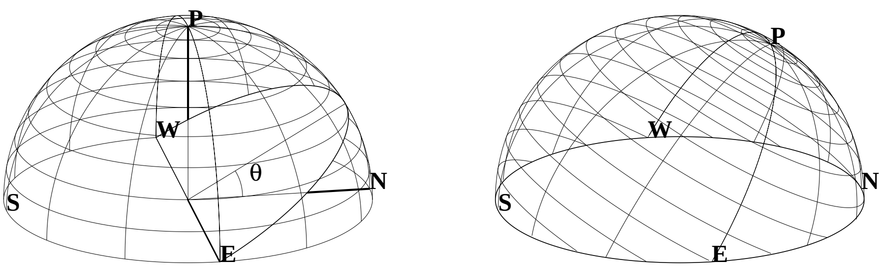
Astronomy and Spherical Geometry and Trigonometry

Our field of vision is the sphere. The sphere parametrizes all directions emanating from a point. Thus the spherical geometry is the one with which we are most intimate. This may appear as a paradoxical fact. First we can experience the sphere in two different ways. Usually we think of it as embedded in 3-dimensional space as a ball we can touch. The great circles, which are intersections of the sphere with planes through the center and play the role of lines, are obviously no straight lines but curved. We can only see half of the sphere, and a large part of that in strong distortion, but we can of course turn the sphere around and get a feeling for it. This is typical for our instinctive knowledge of Euclidean through our ability to move around and touch things. Thus our spatial sense is an integration of many senses, not only the visual. The Earth (which of course is not a perfect sphere, neither locally (uneven terrain) nor globally (an ellipsoid), but that is of minor concern for our present purposes) we cannot literally turn around, but we can imagine it, and we certainly can move around it. But secondly we can also experience a sphere from the inside, which makes it very different. The most tangible manifestation of it is the starry vault above us. Great circles are no longer curved but straight lines, being the intersection of planes through the eye, as being in the center. The celestial sphere is only accessible to us by sight, not by touch, and could be, for all what we know infinitelt distant to us. Thus the Greeks knew about two geometries, the flat on earth (although of course Greek culture was well aware that the earth was 'round') in which we can move around and touch, and which is extended to a flat 3-dimensional space, although our vertical movement is somewhat hampered, and the spherical celestial. The latter can of course be modelled in flat three-dimensional space, and was thus thought of something ultimately reducible to flat three-dimensional geometry. But as noted, the celestial sphere does not need to be embedded as a sphere in space, if so, the natural question is what lies beyond it, but can be seen more abstractly as the space of all directions and situated beyond all points in space, thus at infinity. It is not clear that the Greek did make this possibility explicit.

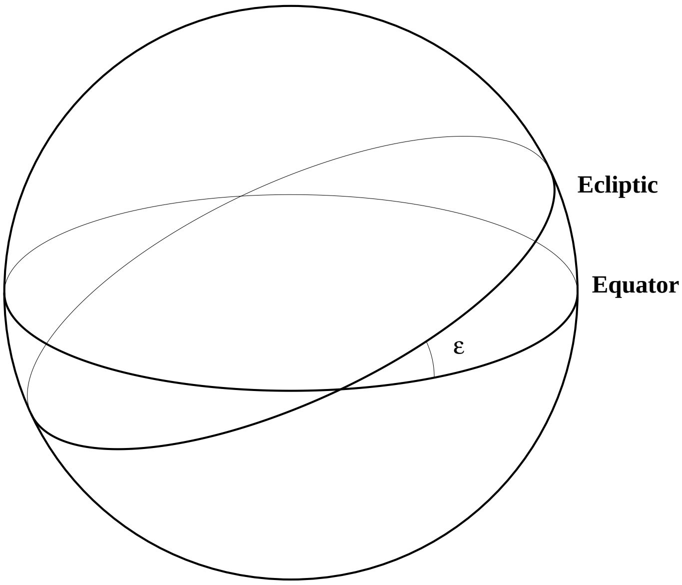
The Greeks did realize at an early stage that the Earth was not flat, something manifested by the different positions of the horizon visavi the celestial sphere. From this they concluded that the Earth was a sphere, this being the most perfect solid. Thus the spherical nature of the celestial vault led to the spherical form of the Earth. But they did not draw the conclusion that the Earth rotated, that contradicted common sense. They had no understanding of inertia and imagined that if this was the case there would be a strong from west to east, which would be the case would the universe be filled with stationary air.

The movement of the fixed stars was easy enough, but that of the Sun and the Moon and the 'wandering stars', the so called planets with their at times retrograde motion was something quite different. Already Plato proposed as a research projects to give a mathematical explanation, not necessarily physically supported, to 'save the appearences', meaning to give a model. To do so they were hampered by the ideological commitment that movements had to be circular, the most perfect of all movements. As straightfoward models did not confirm with observations, of which there were long series stemming from the Babylonians, this had to be modified, by what eventually would be intricate systems of epi-cycles. Thus astronomy provided an intriguing blend of pure mathematics and empirical observation.

One problem that confronts the astronomer is to have a way of assigning positions of the celestial objects, and the solution is the use of spherical coordinates which thus has a long history of a couple of thousand years, and thus predates so called Cartesian coordinates. The simplest thing is of course to look locally at the hemi-sphere above us, bounded by the great circle of the horizon. Two natural measurements present themselves. One is the altitude of the object above the horizon, the other is its direction. The first is godgiven so to speak as we have the horizon as the reference, for the second you need to fix a point on the horizon. In principle there is no canonical choice, however, due to the rotation of the celestial sphere (from a mathematical point of view it does not make any difference in the context) there is a natural plane going through the axis and perpendicular to the horizon (except at the poles of course). It cuts the horizon in two antipodal points, North and South. Which one to take as a reference is a matter of choice. Note that it takes some effort to locate the line. During historical time there has been a fairly bright star close to the north pole - the Polar Star[7] which would have made it easier. Length on the sphere are measured by angles. Those measures have nothing to do with scaling and the units to be used intrinsic to the sphere, and having nothing to do with arbitary units of measurement, as in flat Euclidean space, but refer to fractions of circles, or if you prefer great circles on the sphere[8]. Now the positions of the fixed stars move during the night, to get a more invariant position, we need invariant references, such as the celestial equator.

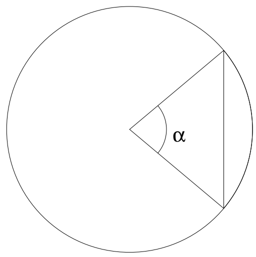


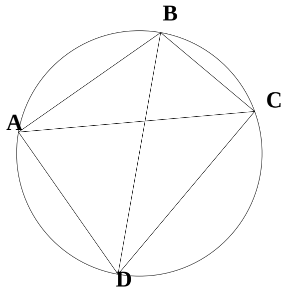
To find a point of references on it, one uses the fact that there is another important great circle on the celestial sphere, namely the ecliptic, traced by the movement of the sun. The ecliptic cuts the celestial equator in two antipodal point, corresponding to the equinoxes. The one corresponding to the vernal equinox is taken as the point of reference. The (signed) distance to the equator is referred to as the declination, and the distance to the reference point as the right ascension counting counter clockwise (positive direction), while the sky rotates in the opposite direction (from east to west when facing south). The sun moves in the the same positive direction (i.e. opposite to that of the sphere) thus the time between two culminations (facing due south) is somewhat longer than 23h 56m, in fact about 1/360 of 24 hours meaning about 4 minutes hence 24 hours[9].



The celestial equator is given by the plane of the equator of the Earth, hence the plane perpendicular to the axis of rotation, while the ecliptic is given by the plane of the orbit of the Earth. The inclination (ϵ) of the two relative to each other is rather easy to measure, one checks the difference between the altitude of the sun when it culminates in the south on midwinter and midsummer, and it varies slightly. It is roughly 23.5^o. More remarkeble though is that the rotating axis of the Earth is not fixed. It performs a slow circular movement around the poles of the ecliptic in about 26'000 years. This amounts to about $50''$ a year, too little to detect by crude instruments over a year, but the Greeks (Hipparchos) were nevertheless able to do so over long intervals of observations[10]. Another irregularity discovered by the Greeks was that the lengths of the seasons are not the same, which would have been the case if the movement of the Sun along the ecliptic had been uniform. The solution was to put the Earth not quite at the center of the orbit of the Sun.

Anyway the situation gave rise to purely mathematical problems having to do with rotations of spheres with respect to each other, which had to do with both spherical geometry as well as with trigonometry, but first (plane) trigonometry had to be developed.





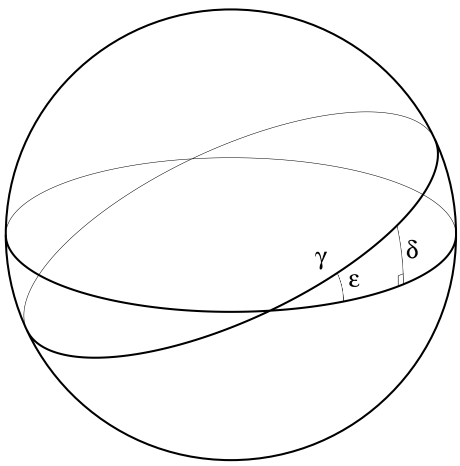
Ptolemy took as his basic definition the length of a chord given by a certain angle. He also normalized the radius of the circle to 60. Thus in modern notation he introduced chord$(\alpha) = 120 \sin \frac{\alpha}{2}$. In order to get systematic tables you needed to obtain addition formulas. For this purpose the following classical theorem comes in handy. Given a quadrilateral inscribed in a circle, denote its vertices by A, B, C, D. Denoting the length of a segment given by X and Y with XY we have the following identity

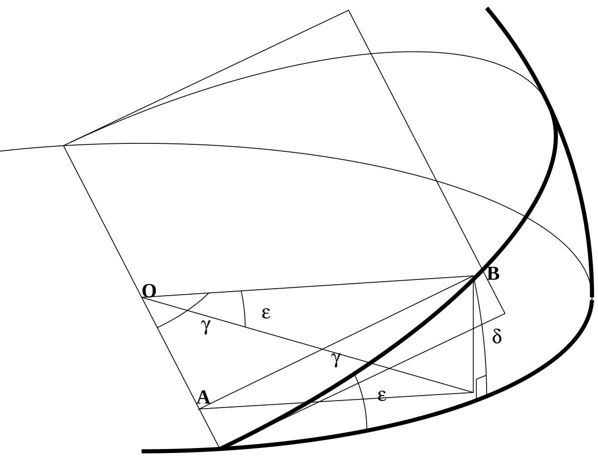
$$AC \cdot BD = AB \cdot CD + AD \cdot BC$$

Which can be translated into the modern addition formulas for sine and cosine[11]. Further more Ptolemy knew how to find the chord of half an angle by solving a quadratic equation, so one sees here the practical value of such exercises, but of course for it to be applied it has to be translated into numerical values

Furthermore Ptolemy knew from the Greek tradition how to construct a regular pentagon[12], in particular how to compute the chord of 72^o and thus that of 12^o as the chord of 60^o is elementary. By halfing we get to $6^o, 3^o$ and $1\frac{1}{2}^o$ degree. Ptolemy would like to go down to $\frac{1}{2}^o$ but he considered the trisection of an angle impossible and resorted to an approximation based on the fact that if $\alpha < \beta$ then chord(β) : chord$(\alpha) < \beta : \alpha$. Comparing with $\frac{3}{4}^o$ and $\frac{3}{2}^o$ he was able to bracket the value of chord(1^o) and then proceed. In the end he produced a table of chords at $\frac{1}{2}^o$ intervals, presented in sexigesimal notation inherited from the Babylonians, hence in fact using the positional system. To put this in a tabular form shows the gradual increase of the values in front of your eyes and you are invited to numerically interpolate between values. For the aid of this Ptolemy provided the thirtieth of each increase to enable a quick computation of additional minutes to the degree. Such a tabular presentation with quick possibilities of interpolation would be in use for serious computations well into the 20th century. Ptolemy was not afraid of getting his hands dirty, he had to in order to achieve his objectives.

Once numerical plane trigonometry was in place one could tackle spherical geometry and trigonometry. This was part of astronomers tools until the middle of the 20th century although not part of regular mathematical instruction. Spherical geometry can be reduced to three-dimensional euclidean geometry, and hence their theorems can be rediscovered. One typical problem is to find out the declination of the sun at a specific time of the year.





Thus given γ (the longitude of the sun in the ecliptic system, equal to zero at the vernal equinox) and the inclination ϵ to figure out the angle δ. We see that the length of AB is $\sin\gamma$ and hence we have that $\sin\delta = \sin\gamma\sin\epsilon$. This is a theorem in spherical trigonometry. To see the larger context, consider a triangle with sides a, b, c and opposite angles A, B, C. Computing the height h we can express it as either $a\sin B = b\sin A$ or $\frac{\sin A}{\sin B} = \frac{a}{b}$. Thus $\frac{\sin A}{\sin C} = \frac{a}{c}$ and $\frac{\sin C}{\sin B} = \frac{c}{b}$. In spherical geometry we have $\frac{\sin A}{\sin B} = \frac{\sin a}{\sin b}$ instead. If one of the angles, say B is a right angle this simplifies to $\sin A = \sin a \sin b$ as in our case.

Other examples are given when you more generally want to convert to one coordinate system to another, especially to the one based on the horizon. When the declination is positive means that the object is above the horizon and hence in principle visible. An object on the sky moves around a circle centered at the poles, from where it interesect the horizon (if ever) you can compute how long it will be above the horizon and hence when it will rise and set. Thus we see that astronomy gives rise to purely mathematical problems, which the Greek could solve.

However the most impressive was the 'saving of appearances' in the words of Plato. To give a mathematical model to predict the motions of the planets (and the Moon). Ptolmey was able to do this with an extremely elaborate system of epicycles upon epicycles giving a quite good approximation. As it turned out to be a dead end is quite another thing, Copernicus, Kepler and Newton affected a total revolution and launched us into the scientific age.

But the general approach of Ptolemy is by no means dead, in fact it provides rather the rule than exception when it comes to quotidian scientific research, providing mathematical models which are fine turned in order to conform better and better with observation. As far as ingenuity, patience and perserverance are concerned the achievment by Ptolemy is indeed very remarkable.

Notes

[1] There are many variations on the theme, the one presented here is the most elaborated from a literary and dramatic point of view.

[2] It is reported that some notable Roman citizen, (in fact by Cicero) actually stumbled over the grave of Archimedes, which by then was in decay. Cicero saw to it that the neglect was discontinued and the monument restored.

[3] In more modern parlance, a parabola is a conic tangent to the line at infinity. The diameters will then be the lines emenating from that point.

[4] The parabola will have the property that given any set of parallels chords, their midpoints will lie on a line, in fact a diameter. From this we conclude that the tangenmt at B is in fact parallel to AC. This fact is trivial to prove by the techniques of so called analytic geometry. Consider lines with slopes k intersecting the parabola $y = x^2$. The solutions x_1, x_2 to the quadric $x^2 = kx + h$ will satisfy $\frac{x_1+x_2}{2} = k/2$ thus the the midpoints of intersection points will lie on the vertical line $x = k/2$.

[5] How do you get such a result? Recall how you sum $1 + a + a^2 + \ldots a^n$. Set the sum equal to S and we get the identity $1 + aS = S + a^{n+1}$ and solve for S (which works out if $a \neq 1$ and from this you get the finite formula above. (Note this trick is what you do when you compute the sum of a periodic decimal, but then applied to an infinite sequence with no tail.) But Archimedes may have proved it directly. Observe that $\frac{1}{4^n}\Delta + \frac{1}{3}\frac{1}{4^n}\Delta = \frac{1}{3}\frac{1}{4^{n-1}}\Delta$ and start from the identity $\Delta + \frac{1}{3}\Delta = \frac{4}{3}\Delta$ and replace the final term on the left with $\frac{1}{4}\Delta + \frac{1}{3}\frac{1}{4}\Delta$ and proceed inductively.

[6] If $1 > r > 0$ we want to show that $r^n \to 0$. Argue that $r^n > r_0 > 0$ and look at the numbers $\frac{1}{r^n}r_0$ we need to show that those go to infinity, assuming that we can find a largest such number less than all r^n and rename it r_0. Then we have for some n that $0 < r_0 < r^n < \frac{1}{r}r_0$ from which follows that $r^{n+1} < r_0$ a contradiction. In modern matehatics we would argue that any subset of reals bounded from below has a biggest upperbound, which we would choose as r_0 this would lead to the conclusion. In the same way if $\frac{1}{r^n}$ is bounded above there would be a smallest number bounding them. Archimedes might instead have argued that $\frac{1}{r}r_0 - r_0 = \delta > 0$ and hence that $\frac{1}{rn+1}r_0 - \frac{1}{r^n}r_0 > \delta$, thus the sequence steps up by at least δ at each step. We now take as an axiom of the reals, referred to as the Archimedan axiom, that for any $\delta > 0$ no matter how small $n\delta$ tends to infinity, i.e. can be made arbitrarily large. Thus in particular no infintesimals.

[7] Actually the Polar Star is getting closer and closer to the North Pole, it will reach its closest around 2100 A.D. At the moment it is 40 minutes from the pole.

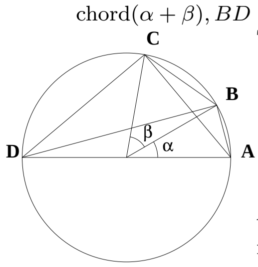
[8] To speak about the Moon as large as a six-pence does not make much sense, you need to specify at which distance it is observed at. As a rule of thumb, a coin one centimeter across held at armlength (60 cm) extends one degree. The convention of dividing the sphere into 360 parts goes back to the Babylonians, who using the sextigesmal system, furthermore divided the degree in 60 minutes, and the minute in 60 seconds, conventions still used in astronomy. One explanation of the choice of 360 is that it is fairly close to the nunber of days in a year, and thus the movement of the sun against the fixed stars is roughly 1^o a day. There is some additional confusion as the Babylonians also divided the horizon, or thus any other great circle of reference in 24 hours, each subdivided in minutes and seconds.

[9] The sun is not a very good clock, only at the average does it keep a 24 hour interval, the deviations are known as the equation of time. Although the diurnal errors are small over the months it can accumulated to around twenty minutes.

[10] The positions of stars are recored with respect to the equatorial axis, dismissing their proper motions, which at most amount to some seconds a year, those positions systematically

shift, and every 50 years new coordinates are published. This is the reason why the positions of stars change visavi the pole, and that only a certain times in history there is a bright star close to the pole. The southern hemisphere does not at the moment have a polar star of any significant brightness

[11] We find $AB = \text{chord}(\alpha), BC = \text{chord}(\beta), CD = \text{chord}(180 - (\alpha + \beta)), DA = 120, AC = \text{chord}(\alpha + \beta), BD = \text{chord}(180 - \alpha)$.



This leads to the formula

$$\text{chord}(\alpha + \beta)\text{chord}(180 - \alpha) = \text{chord}(\alpha)\text{chord}(180 - (\alpha + \beta)) + 120\text{chord}(\beta)$$

Set $\gamma = \alpha + \beta$ and hence $\beta = \gamma - \alpha$ and rearrange the formula above to

$$\text{chord}(\gamma - \alpha) = \frac{1}{120}(\text{chord}(\gamma)\text{chord}(180 - \alpha) - \text{chord}(\alpha)\text{chord}(180 - \gamma))$$

which is, if in disguise, a familiar addition formula for trigonometric functions. The factor of 60 chosen by Ptolemy for the convenience of the numerical values turns out from a mathematical point of view to be a bit cumbersome.

In particular setting $\alpha = -\gamma$ we obtain as $\text{chord}(180-\gamma) = \text{chord}(180+\gamma)$ and $\text{chord}(\gamma) = 60\frac{\text{chord}(2\gamma)}{\text{chord}(180-\gamma)}$ and as $\text{chord}^2(180 - \gamma) + \text{chord}^2(\gamma) = (120)^2$ we can compute the chords of half-angles inductively.

[12]This is a striking construction to be found in the Elements of Euclid. From the modern perspective it is a question of constructing the complex number z such that $z^5 = 1$ and $z \neq 1$ (i.e. a primitive fifth-root of unity.) This amounts to solving the equation $z^4+z^3+z^2+z+1 = 0$. Divide by z^2 and obtain $z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2} = 0$, set $w = z + \frac{1}{z}$ and we translate into the quadratic equation $w^2 + w - 1$ solving the quadratic equation gives you actually the roots $2\cos\frac{2\pi}{5}, 2\cos\frac{4\pi}{5}$ or $\frac{1}{2}(1 \pm \sqrt{5})$ and to get the corresponding sines you need to take further square roots.