Ninth Lecture

Gauss - the Prince of Mathematics[1]

Gauss was born at Brunswick (Braunschweig) in 1777 the son of an uneducated laborer and his wife, who could hardly read. Early on he showed remarkable abilities in mathematics. It is said that even at the tender age of three he spotted his father making a mistake when giving out the renumerations to his staff. At ten he astonished his teacher Büttner by immediately adding an arithmetic series[2]. He then got to be tutored by an older student Barthels, who soon realized he no longer had anything to contribute. His father had no interest in furthering the promising career of his son, but was persuaded to allow it to happen as the duke of Braunschweig took a personal interest in the prodigy and gave him a stipend to study and in the process Gauss also got quite interested in classical languages and philology, which he, to the alarm of his father, considered pursuing, but one thing made him turn to mathematics to which we will return below. He kept a diary with elliptic, not to say laconic, notes from then on for twenty years. Mathematically he was extremly productive, ideas rushed into his head with such force that he only had time to attend to a few, and even those he did not have time to develop properly and publish, thus he anticipated much of 19th century mathematics such as the theory of elliptic functions, hyperbolic geometry, least squares, fast Fourier transform.

Gauss was exceptionally productive and worked in a wide variety of fields, not only mathematics, but also doing empirical as well as theoretical work in Astronomy, Geodesy, Magnetism. It is tempting to compare him with Euler, with whom he shared many characteristics, such as a love and incredible ability to perform numerical computations as well as an omnivorous taste in all kinds of mathematics, as well as physics[3]. But while Euler published almost everything he wrote, Gauss did not, as noted above, have time for that, famous for his motto *pauca sed matura* (few but ripe), he wanted only to present the thoroughly thought out and polished. When it came to computations it is tempting to conclude that Euler did it by brute force like a idiot savant, Gauss could never resist doing it in a clever way finding short-cuts. The difference is telling when it came to celestial calculations, what took Euler three days and made him blind in the process, Gauss did in a matter of hours (saving his eye-sight).

As to his personal life, he had no affection for his father Gebhard, whose honesty and comptence he respected but whose brutality he resented, and thus was rather relieved than sorry when he died while Gauss was still in his early thirties. His mother Dorthea (b. Benze) though, who was always very proud of him and his achievements, lived on in his home until her death at 97 in 1839. Gauss was married twice and widowed twice and the unions brought three plus three children of whom five reached adulthood. With one son - Eugene - he quarreled, referred to him as a 'Taugenicht' (good for nothing). The son early on emigrated to America, where he died at the age just short of 85 in 1896, and showed some remarkable talents, including a prodigious memory and capacity for calculating in his head, as well a mastery of many languages including Sioux[4]. Another son - Joseph - was of a more docile character and assisted his father in survey work.

Gauss was not only a swift worker, but also a hard one, unbelievably diligent. Much of that activity taken up by what we now would consider mindless computations in celestial mechanics and surveys. However, in whatever activity he found himself, he turned it to good use, literally transforming everything to gold that he touched. He took a keen interest in designing telescopes (cf. Newton who made himself his own reflecting telescope) and actually made several very useful inventions such as the heliotrope very useful for surveying by reflecting sunlight visible as a bright star for many miles. Another, even more remarkable invention was the first telegraph in 1833 together with his assistant Weber who exploited the recent discovery of Ørsted as to the connection between electricity and magnetism.

Gauss was fond of literature and read in many languages apart from the Classical ones and his own Native German also in English, French and Danish and after teaching himself Russian in his sixties adding that language as well to his reading repertoire. As a teacher he was very demanding and few students attended his lectures, which he delivered sitting by a table, covered with books, writing in his small and neat handwriting on a small black board put on a stand. He did not want his students to take notes as it impaired their attention. His penetrating gaze out of clear blue eyes was legendary and must have terrified all but the most intrepid of his students. Eisenstein was his favorite one, but he died very early and was not to leave the mark he might have been destined to. Dirichlet and Riemann, both who succeeded him at Göttingen, are his most famous ones[5].

Gauss was by modern standards a bit short $5'1''$ but of stocky muscular build. Of a hypochondrial temperament his health was nevertheless very robust, allowing his demaning working schedule; only during the last year of his life, did his heart start to give up.

The regular 17-gon

As a student Gauss was quite enamoured by classical languages and contemplated a career in philology, but just short of turning nineteen he discovered that the regular 17-gon could be constructed by ruler and compass, meaning that the cosine for the angle $\frac{2\pi}{17}$ could be obtained by a series of quadratic equations (three in fact, leading to a formula involving nested square roots up to three levels). So let us warm up with the regular pentagon the construction of which was known already to the Greeks.

Using complex numbers we can express its five vertices as the solutions to the equation $z^5 - 1$. Those can be expressed by $e^{\frac{2\pi k}{5}}$ for $k = 0, \dots 4$. It has the obvious solution $z = 1$ corresponding to $k = 0$ and factorizing it out the four remaining roots satisfy the equation

$$z^4 + z^3 + z^2 + z + 1 = 0$$

Now set $w = z + \frac{1}{z}$ and note that $\frac{1}{z} = \bar{z}$ and hence that w is twice the real part of z or if $z = e^{\frac{2\pi k}{5}}$ then $w = 2\cos\frac{2\pi k}{5}$. This obviously holds for any regular polygon, given by the solutions to $z^n - 1 = 0$. Now divide the equation with z^2 and get $z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2} = 0$, As $w^2 = z^2 + 2 + \frac{1}{z^2}$ we can rewrite this as $w^2 + w - 1 = 0$ which can be solves explicitly as $w = \frac{-1\pm\sqrt{5}}{2}$. We note that for $z = e^{\frac{2\pi k}{5}}$ $k = \pm 1$ cosine must be positive but for $k = \pm 2$ it has to be negative. Hence $2\cos\frac{2\pi k}{5} = \frac{-1+\sqrt{5}}{2}$ for $k = \pm 1$ while $-\frac{1+\sqrt{5}}{2}$ for $k = \pm 2$. Thus the pentagon can be constructed by ruler and compass. Note also that z can be solved by solving the quadratic equation $z^2 - wz + 1 = 0$ which will of course have two conjugate complex roots, the real part given by the cosine and the imaginary part by the sine of the appropriate angle, after all $e^{\frac{2\pi k}{5}} = \cos\frac{2\pi k}{5} + i\sin\frac{2\pi k}{5}$. Note also that if we double the angle we get a new solution. As $\cos 2\theta = 2\cos^2\theta - 1$ this translates into $w \mapsto w^2 - 2$ and indeed $w + (w^2 - 2) = (w^2 + w - 1) - 1 = -1$ and $w(w^2 - 2) = w(w^2 + w - 1) - (w^2 + w) = -1$

Now set $\omega = e^{\frac{2\pi i}{17}}$. This is a 17th root of unity and all the others are given by ω^n for $n = 0, 1 \dots 16$. Throw away the trivial one $n = 0$ and the remaining satisfy the equation

$$\omega^{16} + \cdots + 1 = 0$$

In modern language this is an equation with Galois group $\mathbb{Z}_{16}$ with a generator T given by $\zeta \mapsto \zeta^3$ for $\zeta = \omega^n$ $(n = 1, \dots, 16)$. In fact if we apply T and its powers to ω we end up with

$$\omega, \omega^3, \omega^9, \omega^{10}, \omega^{13}, \omega^5, \omega^{15}, \omega^{11}, \omega^{16}, \omega^{14}, \omega^8, \omega^7, \omega^4, \omega^{12}, \omega^2, \omega^6$$

Which we can summarize as

$$[1, 3, 9, 10, 13, 5, 15, 11, 16, 14, 8, 7, 4, 12, 2, 6]$$

note that the sum is -1

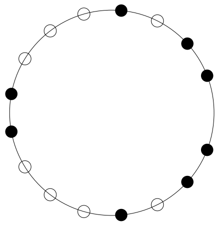
Now split this ordered set into two by taking every other element

$$[1, 9, 13, 15, 16, 8, 4, 2] \quad [3, 10, 5, 11, 14, 7, 12, 6]$$

Let z_1 be the sum of the first eight and z_2 the sum of the second eight. Note that $Tz_1 = z_2$ thus $z_1 + z_2$ and z_1z_2 are left invariant by T and hence are in fact integers. Obviously $z_1 + z_2 = -1$ as to z_1z_2 we get a new sum of ω^n where 1 never appears. As there are 64 terms in the product, and every one of the sixteen terms ω^n $(n \neq 0)$ occurs equally often we see that the sum of the terms will be -4. Why do the powers of ω appear equally often? This actually comes form an important meta-principle which lies at the heart of the whole investigation, namely that of symmetry which provides the clue to all discussions of polynomial equations (and actually beyond). The piint is that this sum is independent of the sixteen possible choices of ω (of course for the running discussion be fix ideas by sticking to a particular value of ω namely the one given initially). Thus z_1, z_2 are the solutions to the quadratic equation

$$z^2 + z - 4 = 0$$

with solutions $\frac{-1\pm\sqrt{17}}{2}$. Which one is z_1? Looking at the figure below where the points of the first sum are dotted black, we see that $z_1 = \frac{\sqrt{17}-1}{2}$ and $z_2 = -\frac{\sqrt{17}+1}{2}$.



Now split up the first sum into two by taking every other thus getting

$[1, 13, 16, 4] \quad [9, 15, 8, 2]$

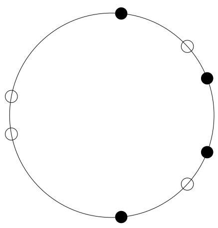
Let the first sum be w_1 and the second w_2. As $w_1 + w_2 = z_1$ and $w_1 w_2$ consists of sixteen distinct terms we find that $w_1 w_2 = -1$ hence they are solutions of the quadratic

$$z^2 - z_1 z - 1 = 0$$

with solutions

$$w = \frac{z_1 \pm \sqrt{1 + \frac{z_1^2}{4}}}{2} = \frac{\sqrt{17} - 1 \pm \sqrt{34 - 2\sqrt{17}}}{4}$$

Which one is which?



We see clearly from the picture on the left that we have $w_1 > w_2$ hence

$$w_1 = \frac{\sqrt{17}-1+\sqrt{34-2\sqrt{17}}}{4}$$
$$w_2 = \frac{\sqrt{17}-1-\sqrt{34-2\sqrt{17}}}{4}$$

Now we continue the splitting up into two groups $[1, 16]$ and $[13, 4]$ their sums are given by $2\cos\frac{2\pi}{17}$ and $2\cos\frac{4\cdot 2\pi}{17}$ respectively and hence totally they add up to w_1, but what about their product?

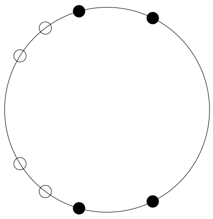
A direct multiplication yields $[3, 5, 14, 12]$ and to get that sum we need to return to $[3, 10, 5, 11, 14, 7, 12, 6]$ which adds up to z_2. Splitting into halves by taking every other one we get the sums u_1, u_2 with $u_1 + u_2 = z_2$ and as before $u_1 u_2 = -1$, hence they are roots to the quadratic equation

$$z^2 - z_2 z - 1 = 0$$

with solutions

$$u = \frac{z_2 \pm \sqrt{4 + z_2^2}}{2} = \frac{-\sqrt{17} - 1 \pm \sqrt{34 + 2\sqrt{17}}}{4}$$

To check which one is u_1 we look at its dotted points



Seeing that $u_1 > u_2$ and hence that $u_1 = \frac{-\sqrt{17}-1+\sqrt{34+2\sqrt{17}}}{4}$

Now we know we have to solve the quadratic

$$z^2 - w_1 z + u_1 = 0$$

with the solutions $x = \frac{w_1}{2} \pm \sqrt{-u_1 + \frac{w_1^2}{4}}$ plugging in the values of w_1, u_1 we obtain

$$\cos\frac{2\pi}{17} = -\frac{1}{16}+\frac{\sqrt{17}}{16}+\frac{\sqrt{34-2\sqrt{17}}}{16}+\frac{\sqrt{68+12\sqrt{17}-16\sqrt{34+2\sqrt{17}}+2(\sqrt{17}-1)\sqrt{34-2\sqrt{17}}}}{16}$$

By noting that

$$\frac{\sqrt{34-2\sqrt{17}}}{\sqrt{34+2\sqrt{17}}} = \frac{\sqrt{17-\sqrt{17}}}{\sqrt{17+\sqrt{17}}} = \frac{17-\sqrt{17}}{(\sqrt{17+\sqrt{17}})(\sqrt{17-\sqrt{17}})} = \frac{17-\sqrt{17}}{4\sqrt{17}} = \frac{\sqrt{17}-1}{4}$$

we may rewrite

$$2(\sqrt{17}-1)\sqrt{34-2\sqrt{17}} = \frac{1}{2}(\sqrt{17}-1)^2\sqrt{34+2\sqrt{17}} = (9-\sqrt{17})\sqrt{34+2\sqrt{17}}$$

while

$$(9-\sqrt{17})\sqrt{34+2\sqrt{17}} = 8\sqrt{34+2\sqrt{17}}+(1-\sqrt{17})\sqrt{34+2\sqrt{17}} = 8\sqrt{34+2\sqrt{17}}-4\sqrt{34+2\sqrt{17}}$$

from which we can, as Gauss did, derive the more elegant expression.

$$\cos\frac{2\pi}{17} = -\frac{1}{16}+\frac{\sqrt{17}}{16}+\frac{\sqrt{34-2\sqrt{17}}}{16}+\frac{\sqrt{17+3\sqrt{17}-2\sqrt{34+2\sqrt{17}}-\sqrt{34-2\sqrt{17}}}}{8}$$

In fact the whole thing works for any prime of the form $F_n = 2^{2^n}+1$ i.e. the Fermat primes $3, 5, 17, 257, 65537$ [6]

More generally one may decompose the polynomial $z^n - 1$ in irreducible factors, the so called cyclotomic polynomials $\Phi_d(z)$ defined inductively by

$$z^n - 1 = \prod_{d|n} \Phi_d(z)$$

. With the exception of $\Phi_1(z) = z-1$ they are all palindromic. We may further note $\Phi_2(z) = z+1, \Phi_3(z) = z^2+z+1, \Phi_4(z) = z^2+1, \Phi_5(z) = z^4+z^3+z^2+z+1, \Phi_6(z) = z^2-z+1$. If d_n denotes the degree of $\Phi_n(z)$ it satisfies $\sum_{k|n} d_k = n$ and thus $d_k = \phi(k)$ where $\phi(n)$ is the Eulerfunction of elements $1 \leq k < n$ relatively prime to n.

We may as a final comment look at the regular 7-gon and 9-gon. In the first case using $w = z + \frac{1}{z}$ we transform $x^6+x^5+\cdots+1 = 0$ into $w^3+w^2-2w-1 = 0$

which is the third degree equation we need to solve in order to find $\cos\frac{2\pi}{7}$. In the second case the cyclotomic polynomial $\Phi_9(z)$. We can find it either by dividing $z^9 + z^8 + \cdots + 1$ by $\Phi_3(z) = z^2 + z + 1$ or by setting $\omega = e^{\frac{2\pi}{9}}$ (or any other primitive 9-th root) and look at its six roots $\omega, \omega^2, \omega^4, \omega^5, \omega^7, \omega^8$. Those are invariant by multiplication by the primitive third roots ω^3, ω^6 hence Φ_6 is a polynomial in z^3, in fact a quadratic such, which can be no other than Φ_3 and thus $\Phi_6 = z^6 + z^3 + 1$. Proceeding as before we end up with the cubic $w^3 - 3w + 1 = 0$.

Disquistiones Arithmeticae

The book was published in the summer of 1801 when Gauss was twenty-four but was written already three years earlier. It contains the results above and is a systematic compendium of elementary number theory and its ramifications and as such it has provided the basics for number theory ever since. More precisely it starts out with a pedestrian exposition of congruences, introducing standard notation, including solving linear congruences by the method of continued fractions, intimately related to the Euclidean algorithm, which goes back to Euler and Lagrange. Then there is a discussion of powers and two proofs of Fermat's little theorem to the effect that $a^{p-1} = 1$ if $a \neq 0(p)$ and the related notion of primitive elements, which also goes back to Euler. This is incidentally related to periods of decimal expansions of rational numbers, which he also discusses. Then there is a discussion of how to compute the Euler totient using its restricted multiplicativity, familar to any student of an elementary text in number theory. The presentation is leisurely and filled with worked out examples, obviously Gauss loved to play around with these and he often prefers to illustrate a general method by a particular example than to formulating it. Things starts to pick up when second degree equations are considered. Crucial is the notion of quadratic reciprocity, which had been conjectured by Euler but first proved by Gauss in this work (other attempts had been made by Legendre). There are a lots of results of an elementary notion that are not usually encountered today. As an example the following theorem: If a is a prime of the form $8n + 1$ there will be some prime number p less than $2\sqrt{a} + 1$ such that a is a non-residue mod p.

The bulk of the work is devoted to quadratic forms, especially binary. Given a quadratic form $ax^2 + 2bxy + cy^2$ with integral coefficients what values can it take for integers x, y? He introduced an equivalence between them by using substitutions effected by elements of $SL(2, \mathbb{Z})$ (as well as elements of determinant -1). Such will not change the discriminant $D = b^2 - ac$ which hence is an invariant, but it is not a complete invariant, many inequivalent forms may have the same invariant, but he showed that there could only be a finite number of classes, out of which he singled out a special one $x^2 - Dy^2$ as the principal (thus any square-free D can occur as a discriminant). Furthermore he introduced an abelian group structure on the classes associated to a given discriminant before such notions became explicit. He noted that there was a fundamental difference between positive and negative discriminant, in the latter case there were only a finite number of solutions to $x^2+Dy^2=m$, while in the positive case $x^2-Dy^2=m$ there is an infinite number if there are any. Classical cases such as primes which are the sum of two squares, or a sum of a square and twice a square are simple special cases of his analysis, and can be described by congruences. In the case of Pell's equation $x^2-Dy^2=1$ he writes down the general solution given a minimal non-trivial one (x_1,y_1) in terms of $x_n+.y_n\sqrt{D}=(x_1+y_1\sqrt{D})^n$. An eight chapter had been projected, but it was not printed to keep down costs. A sequel was planned but never written, after all Gauss was severely distracted by other time consuming work. He did, however, later write on biquadratic reciprocity which necessitated a detour into what is now known as Gaussian integers, namely integers of the form $a+bi$ where a,b are (rational) integers. A ring which has an Euclidean algorithm and hence is a so called principal ideal domain.

Digression on quadratic forms with negative discriminants

Although Gauss did not present his theory in geometric terms, it has been suggested (by Felix Klein) that he nevertheless was familiar with it. But before this let us recall some basic facts about quadratic equations and quadratic fields over the rationals. A quadratic number θ is a number that satisfies an irreducible quadratic polynomial with rational coefficients, i.e. satisfying an equation[7]

$$x^2-px+q=0$$

Given a root θ we can consider all elements of the form $\eta=\alpha\theta+\beta$. They are closed under addition and multiplication, and each element (except 0 of course) has an inverse. They form a field denoted by $\mathbb{Q}(\theta)$. This can be viewed as a vector space of dimension 2 over $\mathbb{Q}$. Any element η in the field satisfies a quadratic equation because the three elements $1,\eta,\eta^2$ cannot all be linear independent. If we insist that the equation is monic (i.e. starting out with x^2) the equation is unique $x^2-Px+Q=0$ where P and Q depend on η. By the theory of quadratic equations, it will have two roots (which cannot coincide because of the condition of irreducibility) and setting the other root η' we will have $P=\eta+\eta', Q=\eta\eta'$[8]. The coefficient P will be called the trace of η denoted by $\mathrm{Tr}(\eta)$ and Q will be called the norm of η and be denoted by $\mathrm{Nm}(\eta)$. The trace is additive, and the norm is multiplicative, both are elementary symmetric functions of η,η'. Another symmetric function is the discriminant $D(\eta)=(\eta-\eta')^2$, which measures the difference between the two roots. Being symmetric it can be expressed in P,Q and indeed[9]

$$(\eta-\eta')^2=(\eta+\eta')^2-4\eta\eta'=P^2-4Q$$

In terms of the discriminant we can write out explicitly the solution to a quadratic equation $x^2-Px+Q=0$ by $x=\frac{P\pm\sqrt{D}}{2}$. By using the properties of the trace and the norm we can compute them for any number η knowing it for θ. In fact

$$\mathrm{Tr}(\eta)=\mathrm{Tr}(\alpha\theta+\beta)=(\alpha\theta+\beta)+(\alpha\bar{\theta}+\beta)=\alpha\mathrm{Tr}(\theta)+2\beta$$

and

$$\mathrm{Nm}(\eta) = \mathrm{Nm}(\alpha\theta + \beta) = (\alpha\theta + \beta)(\alpha\bar{\theta} + \beta) = \alpha^2\mathrm{Nm}(\theta) + \alpha\beta\mathrm{Tr}(\theta) + \beta^2$$

and maybe more interesting

$$D(\eta) = \mathrm{Tr}(\eta)^2 - 4\mathrm{Nm}(\eta) = (\alpha\mathrm{Tr}(\theta) + 2\beta)^2 - 4(\alpha^2\mathrm{Nm}(\theta) + \alpha\beta\mathrm{Tr}(\theta) + \beta^2) = \alpha^2 D(\theta)$$

Thus elements in the quadratic field have the same discriminant up to a square, and that is also sufficient to be in the same field, as the formula for the roots show. This is a crucial observation which tells you right away when two quadratic elements belong to the same field (there is no immediate analogy for fields of higher degrees).

We also have the important notion of integral elements. Those are the ones which satisfy a monic equation with integral coefficients. An equivalent formulation is that an element τ is integral iff the subspace that is generated by integral combinations of $1, \tau$ is closed also under multiplication, i.e. forming a ring. The integral elements of a quadratic form a ring, which is the biggest subring of the field, with rank two over the integers. An arbitrary element η of the field satisfies an equation of form $ax^2 + bx + c$ where (a, b, c) are relatively prime integers. There is a unique such if we fix the sign of a . We have that a is the smallest positive integer such that $a\eta$ is integral. Note that the discriminant of an integral element is obviously an integer. Conversely one may show that if the discriminant is an integer, than the element is integral.

We will now restrict to the case of negative discriminant, then the field (from now on referred to as imaginary quadratic and denoted by $\mathbf{K}$ or $\mathbb{Q}(\theta)$ when a generator θ is specified) can be embedded in $\mathbb{C}$ in such a way that there is a universal conjugation that restricts to the special conjugation for each quadratic subgroup, namely complex conjugation. Thus we find that $\mathrm{Tr}(z) = z + \bar{z}, \mathrm{Nm}(z) = z\bar{z}$ (making the linear and multiplicative character of the trace and norm respectively immediate). Furthermore we find that $D(z) = (z - \bar{z})^2$. By the universality we have that the norm $z\bar{z}$ is a common quadratic form which will play a crucial role in what follows. We also note that any subring R of the imaginary quadratic consisting of integral elements will form a discrete subgroup or rank two. Any additive subgroup of finite rank will be of rank one or two, and if closed under multiplication of some ring with complex member, of rank two. Such additive subgroups of the field will be referred to as fractional ideals. (They are just modules of the ring contained in the field.) If they are contained in the ring they will be ideals. The crucial thing is that fractional ideals may be multiplied and still be fractional ideals. This follows that the product is necessarily a subgroup of finite rank over the integers, and hence of rank two. It follows that R plays the role of the identity in that multiplication. Furthermore to each fractional ideal I we may form the inverse, namely

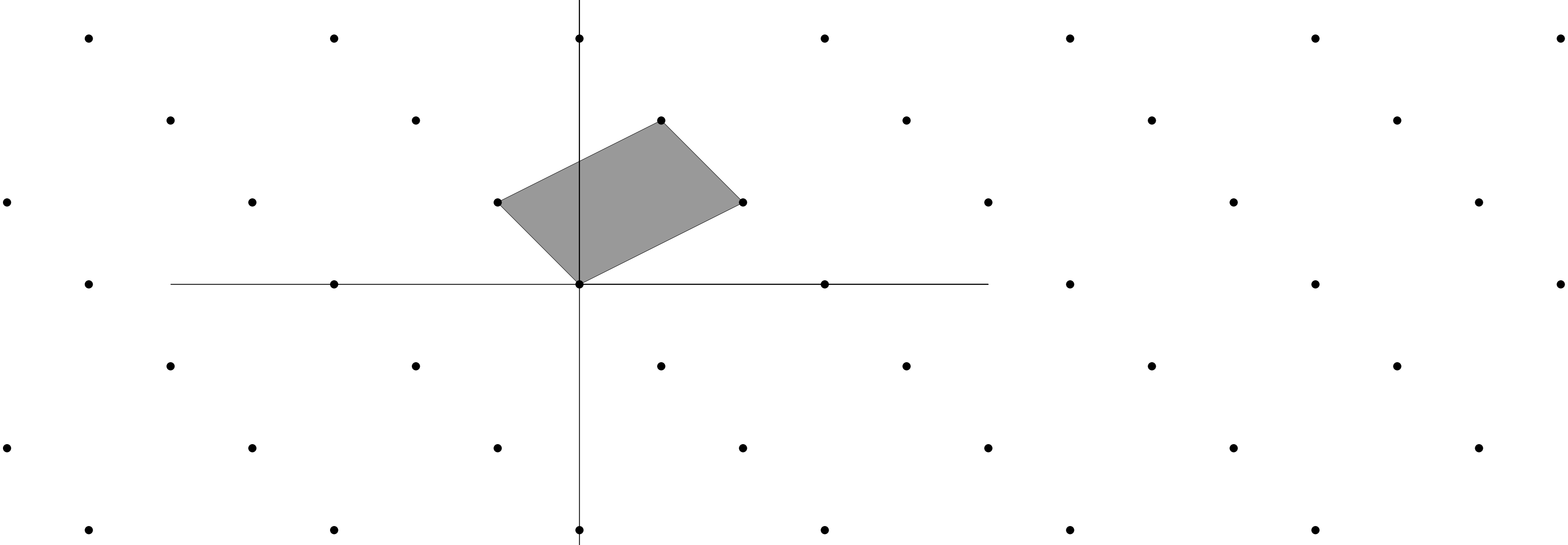
$$I^{-1} = \{a \in \mathbf{K} : ab \in R \quad \forall b \in I\}$$

This is easily seen to be a fractional ideal, and more or less by definition we have $I \cdot I^{-1} = R$. So let us now look at the situation a bit more closely.

The basic concept is thus that of a lattice. A lattice $\Lambda \subset \mathbb{C}$ is a non-dense rank-two additive subgroup of the complex numbers, hence it is generated by two complex numbers ω_1, ω_2 such that $\tau = \frac{\omega_1}{\omega_2}$ is non-real (and if necessary by switching the ωs can be assumed to have positive imaginary part i.e.belonging to the upper half plane $\mathbf{H}$). On $\mathbb{C}$ we have an inner product $< * \cdot * >$ given by $< z \cdot w >= \frac{1}{2}(z\bar{w} + w\bar{z})$ thus note that $\|z\|^2 = |z|^2 = z\bar{z}$. We are now interested in lattices such that $|\omega_1|^2, < \omega_1 \cdot \omega_2 >, |\omega_2|^2$ are all integers (a, b, c)[10], then more generally

$$|m\omega_1 + n\omega_2|^2 = m^2|\omega_1|^2 + 2mn < \omega_1 \cdot \omega_2 > +n^2|\omega_2|^2 = am^2 + 2bmn + cn^2$$

defines an integral quadratic form. The parallelogram spanned by ω_1, ω_2 is called the fundamental parallelogram.



The fundamental parallelogram is not determined by the lattice, but depends on what basis is chosen. A basis change is affected by a matrix of type $M = \begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{pmatrix}$ with $\det M = ac - bd = \pm 1$. Thus equivalent quadratic forms belong to the same lattice. Gauss makes a point of distinguishing between forms which are properly and improperly equivalent depending on whether the determinant is positive or negative, i.e. whether it preserves the orientation of the basis or not. Such transformations preserve the area of the parallelogram (up to sign) and this area can be interpreted as the square root of the discriminant of the quadratic form[11]. We may choose a normalized basis as follows: Pick an element ω_1 with smallest norm, and then ω_2 with next to smallest. As $-\omega_2$ has the same norm we may choose it such that $< \omega \cdot \omega_2 >\geq 0$. Hence that $c \geq b \geq 0, 0 < a \leq c$, such forms are referred to as reduced. We may also classify the shapes of lattices, saying that two lattices Λ, Λ' are similar iff there is λ such that $\lambda\Lambda = \Lambda'$. The shapes are classified by $\tau = \omega_2/\omega_1$ where by choice of ordering we can make sure τ belongs to the upper half plane. And if we take into account the action of $SL(2,\mathbb{Z})$ we get $\tau \mapsto \frac{a\tau+b}{c\tau+d}$. There is also another way we can associate an element of the upper half-plane to a definite quadratic form namely by letting (a,b,c) correspond to the equation $az^2+2bz+c=0$ and pick the root τ_{CM} with positive imaginary part. This will be referred to as the CM-point of the quadratic form. Now those turn out to be essentially the same, in fact setting $\tau = \frac{\omega_2}{\omega_1}$ we obtain

$$\tau + \bar{\tau} = 2\frac{< \omega_1 \cdot \omega_2 >}{|\omega_1|^2}$$

and

$$\tau\bar{\tau} = \frac{|\omega_2|^2}{|\omega_1|^2}$$

hence τ satisfies the quadratic equation $az^2 - 2bz + c = 0$ and hence $\tau = -\tau_{CM}$[12]

Now given a lattice Λ we may consider the complex numbers z such that $z\Lambda \subset \Lambda$. Trivial examples are z an integer, but can there be other complex numbers? Obviously we are on the look out for lattices that can appear as fractional ideals for some order. The condition is that

$$\begin{aligned} z\omega_1 &= a\omega_1 + b\omega_2 \\ z\omega_2 &= c\omega_1 + d\omega_2 \end{aligned}$$

Thus we see that z has to be an eigenvalue of a matrix $M = \begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{pmatrix}$ with integral coefficients. This means that z satisfies a quadratic equation $z^2 - Tz + N = 0$ where $T = \mathrm{Tr}(M) = z + \bar{z} = a + d$ and $N = \mathrm{Nm}(M) = z\bar{z} = ad - bc$. Those complex numbers obviously form a ring, and in fact a lattice generated by $< 1, \tau >$ for some τ where τ satisfies some integral quadratic equation[13]. Conversely given such an element z along with its matrix M we can find the eigenvectors. They will all be multiples of $(1, \frac{z-a}{b})$ thus elements of $\mathbb{Q}(\tau)$. By multiplying it with suitable ω_1 we can get a lattice generated by ω_1, ω_2 corresponding to an integral form (A,B,C). In fact we would have

$$\begin{aligned} |\omega_1|^2 = &\quad A \\ < \omega_1, \tfrac{z-a}{b} > = &\quad \tfrac{T-2a}{b}|\omega_1|^2 = \quad B \\ |\tfrac{z-a}{b}|^2|\omega_1|^2 = &\quad C \end{aligned}$$

From which we conclude that

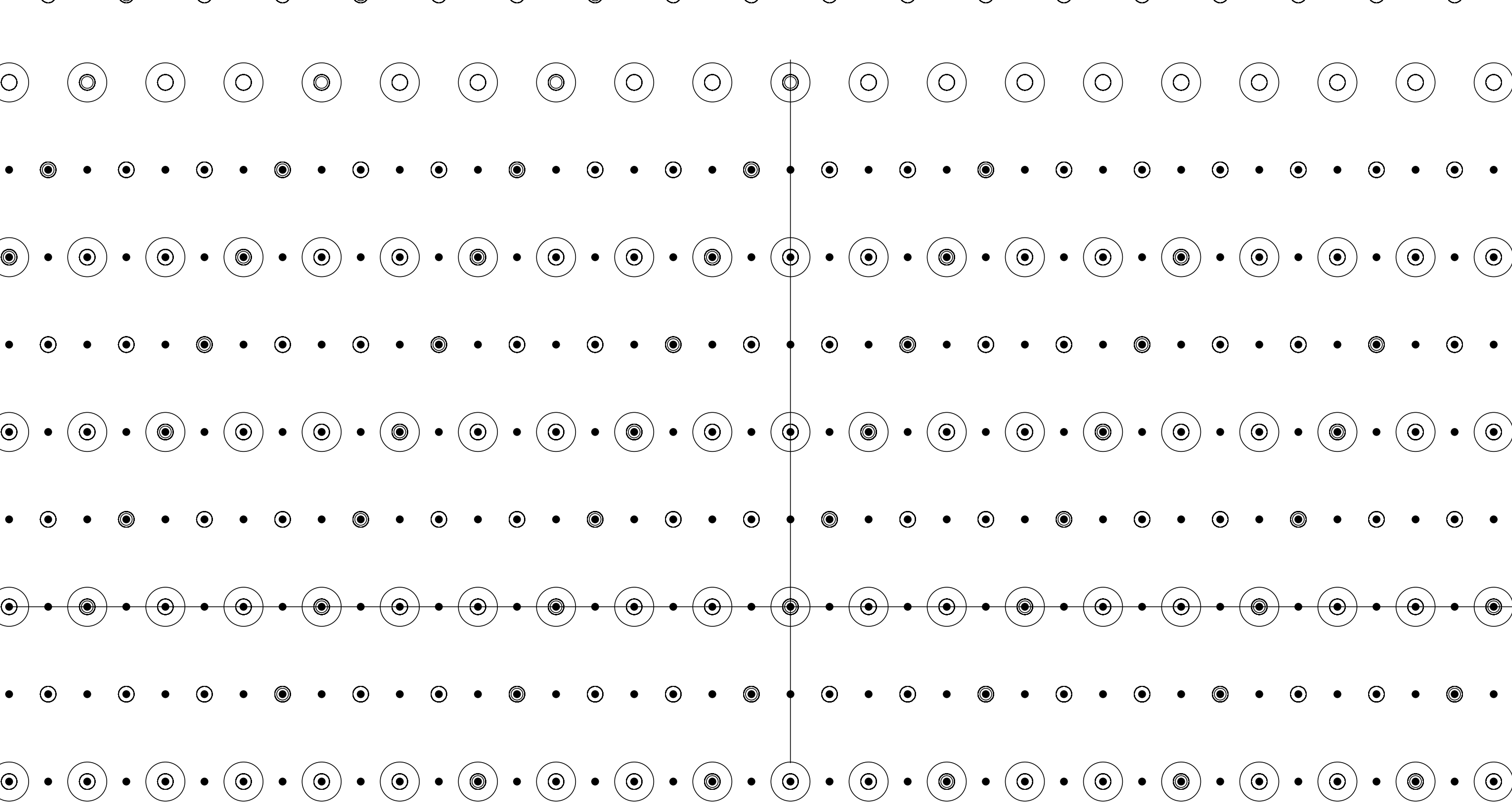
$$\begin{aligned} \tfrac{B}{A} &= \tfrac{d-a}{b} \\ \tfrac{C}{A} &= \tfrac{N-aT+a^2}{b^2} \end{aligned}$$

This allows us to compute the discriminants of the CM point and that of the element τ above and conclude that they are equal as expected.

The problem is to find the ω_1 given τ_{CM}. It is equivalent to finding an element ω_1 in the ring such that $\mathrm{Nm}(\omega_1) = A$, and as not all integral values are taken as norms, the problem is non-trivial[14].

Thus to each quadratic form $Am^2 + 2Bmn = Cn^2$ we may first associate a quadratic element τ_{CM} which defines a field $\mathbb{Q}(\tau_{CM}) = \mathbb{Q}(\sqrt{D}), D = B^2 - AC$ (the reduced discriminant). Then we can consider all lattices spanned by ω_1, ω_2 such that $\frac{\omega_2}{\omega_1} = -\tau_{CM} = \tau$. One such example is given by $< 1, -\tau_{CM} >$. To that we may associate a ring $R = R(\tau)$ its ring of Endomorphisms [15], for which it is a module. This ring will be generated by $1, z$ where $z \cdot 1 = a + b\tau, z\tau = c + d\tau$. Working it out we get that $b\tau^2 + (a - d)\tau - c = 0$. Hence we should choose $b = A, a - d = 2B, -c = C$, thus the ring R is generated by $1, A\tau$ and $A\tau$ corresponds to the matrix $\begin{pmatrix} 0 & 1 \\ \text{-C} & \text{-2B} \end{pmatrix}$. The discriminant of $A\tau$ will be $4(B^2 - AC)$ which up to the factor 4 is the discriminant of the quadratic form. The module $< 1, -\tau_{CM} >$ corresponds to the form $m^2 - \frac{2B}{A}mn + \frac{C}{A}n^2$ and by choosing $\omega_1 \in R$ such that its norm is A we get a lattice Λ spanned by $\omega_1, \omega_2 = \tau\omega_1$ which recaptures the original form and is also an ideal of R. Thus if we have two quadratic forms with the same discriminant, they will be ideals to the same ring R and hence can be multiplied giving rise to a new ideal in R which will correspond to a quadratic form which by definition will be the one defined by Gauss as a product. Gauss considered as special cases, quadratic forms of type $m^2 + Dn^2$ with discriminant $-D$ and D square free. They are associated to the complex numbers $\tau = i\sqrt{D}$ (i.e. solutions to $\tau^2 = -D$). They form rings as the corresponding lattices Λ satisfy $\Lambda^2 \subset \Lambda$. And they will act as the identity for the group associated with a fixed discriminant. The inverses of ideals will only exceptionally be ideals and hence not directly associated to (integral) quadratic forms, but by suitable scaling they will become so.

Now after those preliminaries the situation becomes really interesting. It turns out the group to consider is the group of fractional ideals, modulo principal ones, i.e. ideals generated by a single element[16]. If the ring R has unique factorization (such as $\mathbb{Q}[i]$) all ideals are principal and the group is trivial. Generally the order of the group will be equal to the number of inequivalent (under the action of the group $\mathrm{SL}(2, \mathbb{Z})$ of integral matrices of determinant 1) forms with the same discriminant. The number is referred to as the class number. It can also be calculated by considering a fundamental domain for the action of $\mathrm{SL}(2, \mathbb{Z})$ on the upper halfplane considered above[17] a classical one given by $\{z : \mathrm{Tr}(z) \leq 1, \mathrm{Nm}(z) \geq 1\}$. In that region we can work out the number of CM-points with discriminant D.



The example of $\mathbb{Q}[\sqrt{-5}]$

Above we present the lattice $\mathbb{Q}[\sqrt{-5}](R)$ with discriminant 5 ($\tau_{CM} = \sqrt{5}i$) and area $\sqrt{5}$ of a fundamental parallelogram, as well as the sub-lattice the principal ideal $\mathcal{I} = (1+\sqrt{5}i)R)$ whose discriminant will be $6 \cdot 5 = 30$. The corresponding quadratic form will be $6m^2 + 30n^2$ which is not primitive, the primitive form will correspond to the original lattice R. The quotient $R/(1+\sqrt{5}i)$ will be cyclic of order 6 and we can add the element 2 to $\mathcal{I}$ and get an ideal $\mathcal{J}$ of discriminant -20, which is not principal[18]. A fundamental parallelogram will be spanned by $2, 1+\sqrt{5}i$ and the corresponding quadratic form will be $4m^2 + 4mn + 6n^2$ with discriminant -20. The ideal $\mathcal{J}^2$ will be spanned by $2, 2(1+\sqrt{5}i)$ and have discriminant -80.

The presentation above is of course not, as already remarked, found in Gauss work, but all the ideas exist there and were to provide inspiration for the algebraic concepts related to rings and ideals that would be developed during the 19th century, especially as it relates to algebraic number theory. Gauss came up with estimates of class numbers and it was only in the late 60's when the classification of all negative discriminants with class number 1 was completed. The short list is given by

$$D = -1, -2, -3, -7, -11, -19, -43, -67, -163$$

The Fundamental theorem of Algebra

A modern proof of the fundamental theorem of algebra is very simple using the elements of elementary complex function theory. Given any polynomial $P(z)$ it is easy to see that $\lim_{|z|\mapsto\infty} |P(z)| = \infty$ by noting that the leading term will be dominating for large $|z|$. Thus in particular if $P(z)$ never vanishes then $1/P(z)$ becomes a bounded entire function which by Liouville must be constant. That theorem comes more or less directly from the Cauchy integral formula. In fact we can make a Fourier expansion of an analytic function $f(re^{i\theta}) = \sum_{n\geq 0} a_n r^n e^{in\theta}$ and hence $a_n r^n = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} f(re^{i\theta})d\theta$. If the function is bounded by M we obtain the estimate $|a_n| \leq \frac{M}{r^n}$. this being true for all r implies that $a_n = 0 \quad \forall n > 0$ and we are done.

Gauss proved this theorem in his thesis (submitted to the University at nearby Helmstedt) and would during his life produce a number of proofs, none as slick as the one above, although complex analysis of some form is inevitable. Gauss was also a pioneer of the fundamentals of complex analysis anticipating Cauchy, but kept most of it to himself. In particular he was very clear about the geometric representation of complex number (now usually referred to as Argand diagrams) and showed great impatience at mathematicians who abhorred them as mysterious specious.

Celestial Mechanics and Ceres

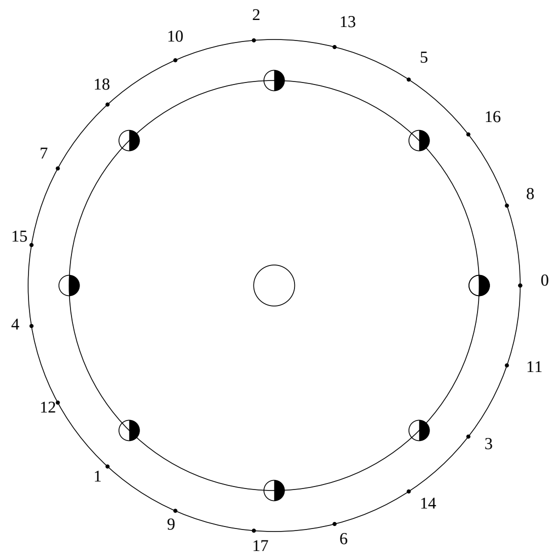
In 1781 William Herschel discovered a new planet - later called Uranus - which was a sensation, as the number of planets had been assumed fixed. Uranus was a transsaturnian planet. A few years later, in fact on January 1 1801, the Italian astronomer Piazzi discovered another one which was named Ceres, a welcome addition as there had been, according to the empirical law by Bode a gap between Mars and Jupiter but Ceres seemed to fill it[19]. Only a few observations were made before the planet disappeared behind the sun. After that the planet was lost, but Gauss figured out a way of making the most of the few observations to determine its path. In so doing the planet was then rediscovered later according to Gauss' predictions. This made Gauss famous and started his association with astronomy and celestial mechanics and in particular funds were sought given to him for an observatory, which would be built in Göttingen. One may speculate as to his motivations. True, Gauss loved to calculate, as did Euler, but one suspects, as noted in the introduction, that while Euler did it by brute force, Gauss went about it in a clever way not being able to help himself. Celestial Mechanics had been worked out extensively by Laplace, but his investigations were theoretical in the sense of not being feasible for actual numerical work, thus Gauss had to approach the subject from scratch developing a lot of tricks and methods he used with great ingenuity, as well as developing the theory of least squares fundamental to observational work [20] and hitting on fast Fourier transform as a computational tool[21] Celestial Mechanics is not an exact science in the sense that in actual work you need to do a lot of approximations and for that purpose avail yourself of a lot of tricks, thus more of an art or handicraft than a conceptual science. Although he did have a deep influence on celestial work any actual presentation of it tends to be a bit *ad hoc* and lack the beauty and simplicity that is the hallmark of pure mathematics at its best. Thus the work on Ceres was just the beginning, later on he would struggle with new asteroids as a steady stream of them were being discovered, especially Juno and Pallas. The latter even stymied him and in his Nachlaß the penciled remark *Lieber der Tod als ein solches Leben* was later found[22]. It was generally felt at the time that it was a waste of Gauss' unique talents to have them squandered on purely observational work and computations.

Easter Formula

As an elementary example of combining simple congruence counting with astronomy, one can consider the date of Easter, for which Gauss provided a formula. To come up with it you do not need the genius of a Gauss, but anyway it can serve as a distraction[23].

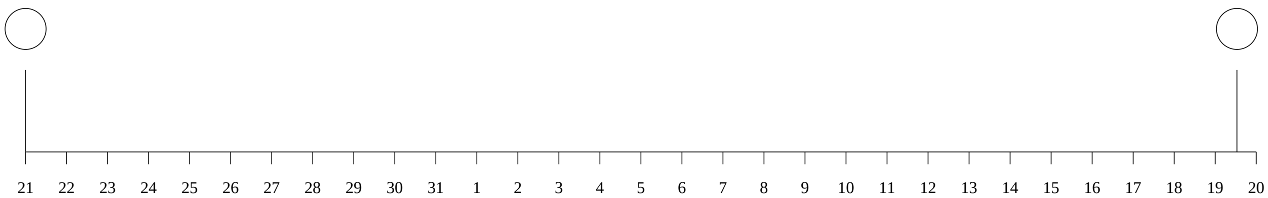
A crucial event of the year is the vernal equinox. It is the time when the ecliptic (the plane of the orbit of the Earth) crosses the celestial equator (the plane of the equator, or equivalently the plane perpendicular to the axis of rotation). At the time night and day is equally long. The time difference between two such events is 365.24 days (known as the tropical year the approximation of which is the basis of the Gregorian calendar, while the Julian makes do with the cruder but simpler approximation 365.25 thus there being 1461 days in a four year cycle). Now the date of Easter Sunday was decided to be on the first Sunday after the first full Moon after the vernal Equinix. Thus Easter Sunday can fall at any time between March 22 (The vernal equinox occuring on March 21) and April 25[24].

First we need to compare with the cycle of the phases of the Moon. The Synodic period, i.e. the interval between two Full Moons is 29.53 days twelve times this is 354.36 which is somewhat short of a full year. However if we take 19 years it will to high accuracy correspond to 235 Synodic periods, and everything will start over again[25]. Of course there is no exact relation, over extended times there will be a drift, and besides the Synodic periods vary slightly due to irregularities of the Moon's orbit (see endnote below). Thus to assume it be exact means Easter will follow an artificial Moon but not the physical. However, by the time a discrepancy will occur, there will be a slight readjustment or mankind, or whatever has succeeded it, will not care about Easter.

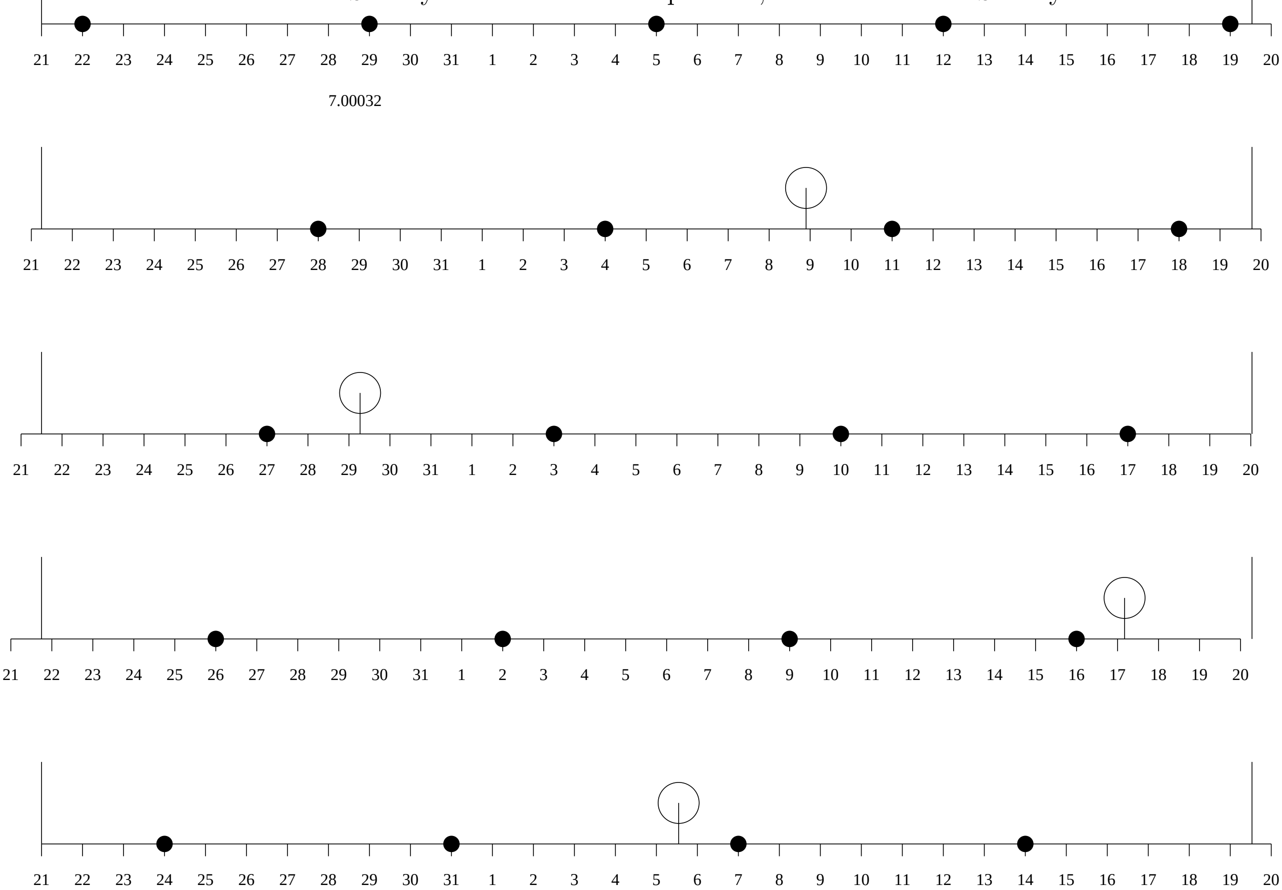


The Synodic period will hence be divided into 19 parts each the length of 1.55421 days of which there will be 7 in the discrepancy between twelve Synodic periods and one year. Thus after the completion of one year there will be a 7/19 th advancement of the phase cycle of the Moon and the Full Moon will hence have occurred at position $-7/19$ or equivalently 12/19 in the phase period. Thus if the Full Moon occurred at position 0 one year later it will occur in position 12/19 and so on. Thus in the figure above the subsequent occurrences of the Full Moon will be noted.

Assume now that at some reference year the Full Moon occurs on the day of the Vernal equinox at a leap year. We can then compare the Synodic period with actual dates going from March 21 to April 19 plus half a day as below.



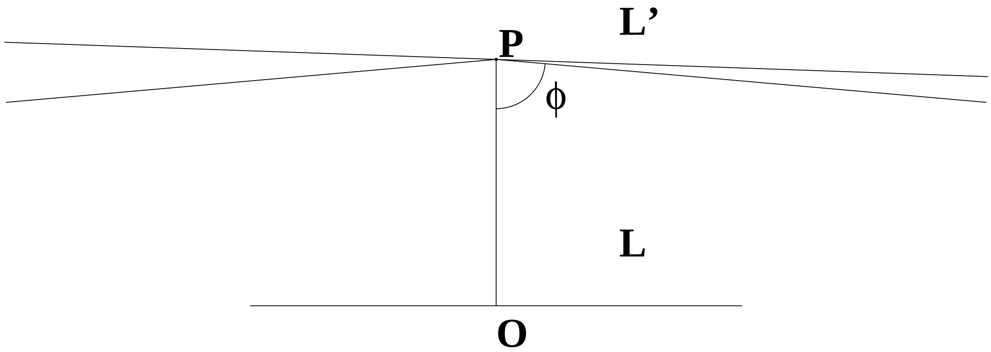
After one year the date scale is shifted back one quarter day as 365 days is about a quarter of a day short of a full tropical year. After four years, a leap day is added to the year to catch up. Below we list four subsequent years along with the positions of the Full Moon. We also set March 22 a Sunday, so the Easter Sunday will be the earliest possible, and mark all the Sundays as well



We can then read off the subsequent dates for Easter Sunday, namely April 11, April 3, April 23, April 7. One may then continue this by hand (as it was classically done) and work out subsequent dates. As we see there will be a number of cycles, in addition of one of length nineteen there will be one of length seven giving the week days, which with the every four leap years combine into a cycle of twenty-eight. Thus the whole thing will repeat itself in $19{\cdot}28 = 532$ years when the Julian calendar is concerned. We note that there are only about 30 possible dates for Easter Sunday, so every date occurs many times, in particular knowing the date for a certain Easter does not help us in determining the date of the next, although one may of course restrict it to a certain narrower interval. Now what Gauss did was to write down a formula in terms of congruences for determining the date for any year. The solution will not be given but offered as a challenge to the reader. It is hard enough for the Julian case, the Gregorian case adds a further non-trivial complication. Gauss provided formulas for both. The Julian Calendar is still uses in Eastern Orthodox churches. Finally the Full Moon at its height occurs on a certain time moment, and hence one also needs to fix the time zone in order to determine the day of the Full Moon. Whether it appears late on a Saturday or early the next morning can make a difference of seven days[26].

Non-Euclidean Geometry

As we have seen there were many mathematicians such as Legendre, Lambert and Saccheri who tried to prove the Fifth postulate of Euclid by trying to get a contradiction. No real logical contradiction was achieved, however many seemingly absurd results[27]. Now absurdity is not *a priori* a reason for rejection, and eventually people came to the understanding that it might be possible to create a logically consistent geometry by negating the postulate, which can be done in two ways. Either, using the reformulation of Playfair, that there were no parallel lines, or that through a point an infinite number of lines could be drawn parallel, meaning non-intersecting, with a given. Janos Bolyai (1802-1860) was the son of a Hungarian student friend of Gauss and who himself had tried in vain to prove the axiom and gave as an advice to his son to stay away from it. This proved only to be a further incentive and the son eventually came to the conclusion that it would be impossible and started to work out the consequences of such an assumption accepting the apparent absurdities as they presented themselves. He showed his results to his father, who relayed them to his friend. Gauss replied that he was unable to praise the work, because doing so would mean praising himself, as he had already worked it out in his youth, but been reluctant to publish it because of the outcry it no doubt would cause[28]. Bolyai was very discouraged by this and more or less dropped out of mathematics, and the work he had carefully prepared between 1820 and 1823 was eventually published as an appendix to a textbook by his father almost ten years later, but by then Lobachevsky had already published his, but that was something Bolyai only would find out almost twenty years later much to his chagrin. Thus to Lobachevsky (1792-56) belongs the honor of priority being out of Gauss' circle he was not inhibited from going public. Lobachevsky called this new geometry, which unlike spherical geometry had no precedence, astral; and thought of it as having definite physical implications. Was that the geometry of the Cosmos? Recall that given a point P outside a line L we may consider the limiting angle for lines L' not intersecting L



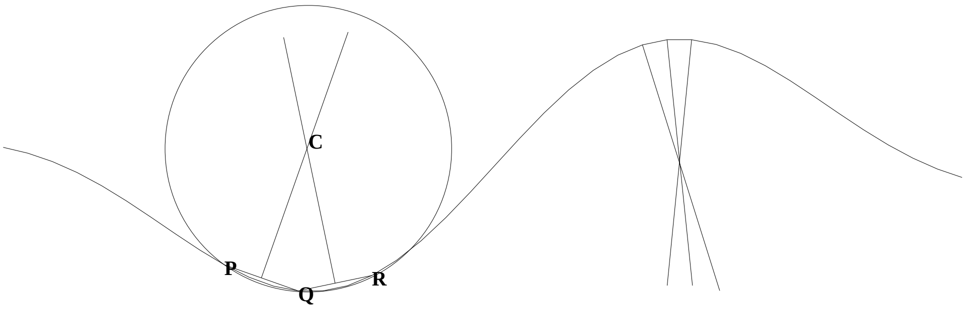
The larger the distance OP the smaller the angle θ and the greater its deviance from $\frac{\pi}{2}$. In Euclidean space a line will extend π in our field of vision (unless we are sighting it in the same direction in which case it will just be a point) regardless of how far away we are. Not so in hyperbolic space, when it will extend 2θ the further away the smaller (which incidentally gives an intrinsic way of measuring lengths). We can also think of it as the angular displacement in direction when we look at an object and move, known as parallax. In Euclidean space objects infinitely far away will show no parallax, not so in Hyperbolic space, where there will always be a parallax, even for objects infinitely far away. At the time no parallax had been obtained for any stars[29] which meant that physical space, even if hyperbolic, would be very close to Euclidean space even when astronomical distances were considered.

The most striking thing of hyperbolic space is that the circumference and area of a a circle increases exponentially with the radius. In fact they are given by $2\pi \sinh r$ and $\pi(\cosh r - 1)$ respectively[30]. For large circles most of the area is concentrated close to the border, and if you want to go to one border point to another along a geodesic, i.e. a line, you will get back close to the origin.

Surveying and Differential Geometry

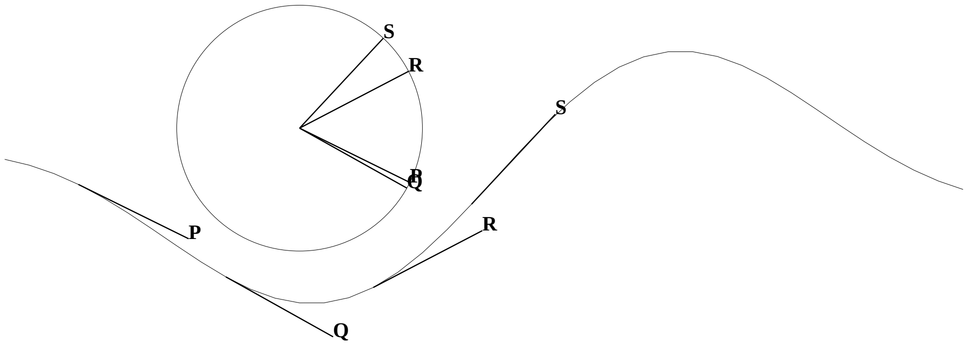
Gauss was for a long time engaged to make a land survey of the principality of Hanover (in a royal union with Great Britain) to connect with a concomitant effort by the Danes. This fostered an interest in the geometry of curved surfaces. But let us start from the beginning.

The notion of curvature of a curve is ancient, and can be seen as a change of change, and thus closely related to second derivatives. First in the 17th century did it get a more precise meaning. Just as a tangent line can be seen as the line that best fits a curve, but also seen as a line that intersects the curve in two coinciding points, we can ask the same thing about circles. What is the best fitted circle, a so called osculating circle, or a circle that intersects the curve in three points? Through three points there is a unique circle (except if they are collinear, in which case it is a line, or a circle with an infinite radius). By letting the points come together we may hope for a limiting circle, whose radius will be called the radius of curvature its inverse the curvature, and whose center the center of curvature.



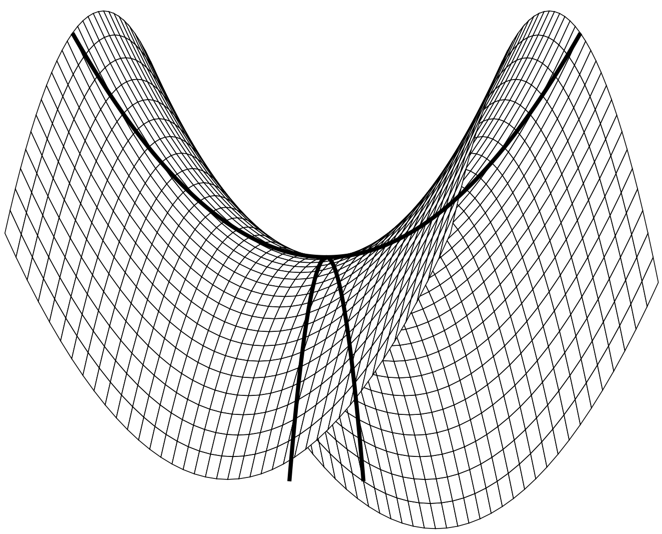
If the points are P, Q, R the approximate center C of curvature may be thought of as the intersections of the midpoint normals of PQ and PR. We may also think of this as the infitesimal intersections of normals. Given a curve at each point P we can think of the line orthogonal to the tangent at P which is the normal. Thus a curve gives rise to a family of normals. Given a normal N_P at P look at nearby normals at P' and their intersections with N_p and the limit when $P' \to P$. Thus to each curve we can associate its involute, the locus of points given by the centers of curvature of the given curve. This was something studied by 18th century mathematicians among them the Bernoullis. Note that circles by this definition have constant curvature $\frac{1}{R}$ where R is the radius of the circle. The involute in this case degenerates to a single point, the center of the circle.

Now consider a curve $y = f(x)$ with horizontal tanget at the origin. It can locally be given as $y = ax^2 + \dots$ where $a = \frac{1}{2}f''(0)$. If we have a circle of radius R tangent at the same point, say $x^2 + (y-R)^2 = R^2$ we can write locally $y = R - \sqrt{R^2 - x^2} = \frac{x^2}{2R} + \dots$. By comparison we find the best fit if we chose R such that $\frac{1}{2R} = a$ i.e. $R = \frac{1}{2a}$ or the curvature $\frac{1}{R} = 2a$. Thus if the change of the change (the second derivative) is big, the radius of curvature is small, and hence the curvature is big.



We can also see the curvature by considering directly how much the direction of a tangent (or equivalently a normal changes) as we move along the curve. The angular change $d\theta$ along a distance ds of the curve gives rise to a rate of change of $\frac{d\theta}{ds}$. In the case of a circle this change is constant, in fact if the radius is R under a complete revolution we have travelled $2\pi R$ and undergone a change of angle given by 2π hence the curvature is $\frac{1}{R}$ as expected. Note also that we can think of curvatures with signs, depending on what side of the curve the centers lie on, or whether the change of angle is positive or negative

When it comes to a surface S in R^3 we may consider the planes through the normal at a point P. They give rise to planar sections, each with its curvature at P. Those curvatures will in general vary by the direction, and there will be two directions of extremal curvatures (taking into account the signs). Those directions are called the principal curvatures, and the directions are perpendicular. This was known to Euler.



Example: Consider the saddle-shaped surface S given by $z = x^2 - y^2$. The normal at the origin is given by the $z-$axis. The planes containing the normal have equations $ax+by = 0$ and there intersections with S are given by parabolas $z = \frac{b^2-a^2}{a^2+b^2}u^2$. The curvatures vary between $\frac{1}{2} \quad (a = 0)$ and $-\frac{1}{2} \quad (b = 0)$ this gives two orthogonal directions. Note also that for $a = \pm b$ the curvatures are zero, as the parabola degenerates into $z = 0$.

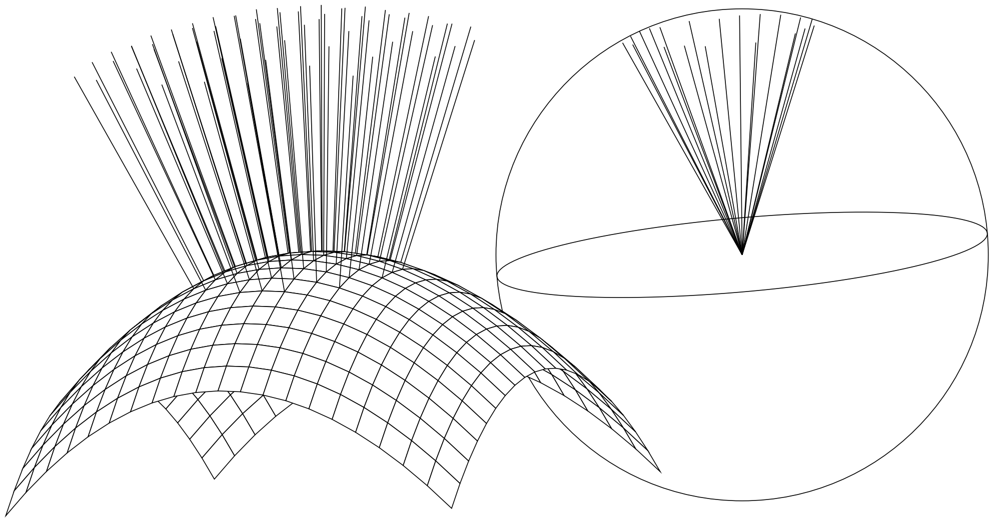
More generally given $z = ax^2 + 2bxy + cy^2 + \dots$ we may look at the plane with angle θ we then get $z = (a\cos^2\theta + 2b\cos\theta\sin\theta + c\sin^2\theta)u^2 + \dots$ the terms $ax^2 + cy^2$ can be rewritten as $\frac{a+c}{2}(x^2 + y^2) + \frac{a-c}{2}(x^2 - y^2)$ hence we can write $z = (\frac{a+c}{2} + \frac{a-c}{2}\cos 2\theta + b\sin 2\theta)u^2 + \dots$ or $z = (A + B\cos(2\theta + \theta_0))u^2 + \dots$ for suitable A, B, θ_0 namely $A = \frac{a+c}{2}, B = \sqrt{\frac{(a-c)^2}{4} + b^2}$ and $\cos\theta_0 = \frac{a-c}{B}$. From this we see that the max $(A+B)$ and min $(A-B)$ are taken when $2\theta + \theta_0 = \pm\pi$ hence at orthogonal angles. This was known by Euler.

Now it is customary to speak about the mean curvature as the sum of the principal curvatures (due to Sophie Germain), in the above case zero, and the total curvature, or the gaussian curvature, as their products, in this case -4. More generally note that the total curvature will be

$$4(A^2 - B^2) = 4(ac - b^2)$$

which fits well. The right hand side is positive when the form is definite, and then the surface lies on one side of the tangent plane, and negative when it is indefinite. When zero it degenerates to zero curvature. Thus we see that for a convex body, the curvature of its boundary is non-negative.

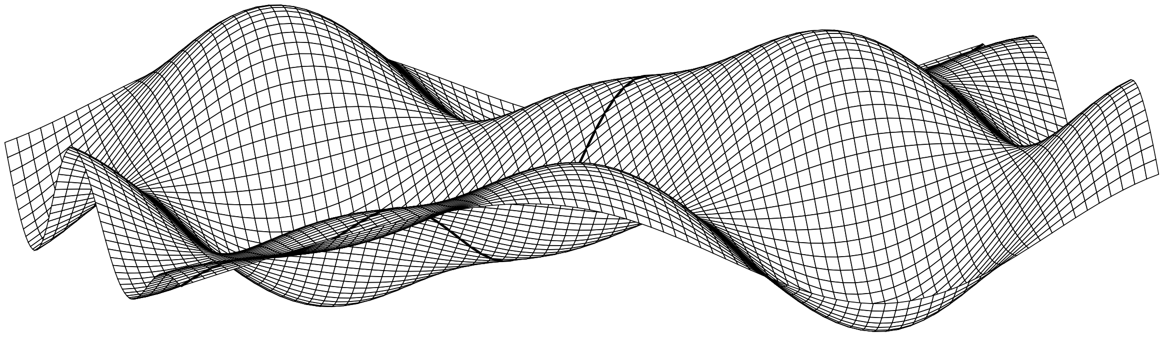
It is one thing to make up a definition, quite another thing to hit upon a fruitful definition, and this turns out to be that in this case. This has to do with Gauss' crucial idea to think of an intrinsic geometry of the surface. To that concept belongs notions such as geodesics, locally the shortest path between two points. Thus given a surface in three-space there are two kinds of curvatures, one merely accidental having to do how the surface happens to be bent (extrinsic) and one intrinsic having to do with the surface itself. An observer constrained to the surface would have no idea of the former only of the latter. A cylinder is curved but it can be folded out flat, using no stretching (nor any tearing except at a meridan along which it is cut), with the geodesics remaining geodesics, while a sphere cannot be flattend, if no stretching allowed then it will invariable burst, and a saddle surface when flattened out will crumple (the opposite problem will appear when trying to wrap them both in paper).



Another definition of of the gaussian curvature is given by the Gauss map. Consider the unit normals to a small neighborhood U of a point P. They will map U to an area U' on the unit sphere. The larger the quotient $A(U')/A(U)$ is the more the surface bends. The gaussian curvature at a point P is simply the limit of this when U shrinks to P (note the analogy with the second definition of the curvature of a curve). The remarkable thing is not so much that those two definitions are equivalent, but that it is invariant under isometric imbeddings of a surface (i.e. one which does not change distances). Thus there should be an intrinsic definition of curvature, which does not depend on a particular embedding. One way is to look at how the circumferences and areas of circles depend on their radii. On a sphere of radius 1 a circle of radius r will be the same as a Euclidean circle of radius $\sin r$ i.e. $2\pi \sin r$, while its area (by Archimedes) will be given by $2\pi(1-\cos r)$. For small r we can write $2\pi(r - \frac{r^3}{6} + \dots)$ and $\pi(r^2 - \frac{r^4}{12} + \dots)$ From this we are motivated to make the definition

$$K = 12 \lim_{r \to 0} \frac{A_0(r) - A(r)}{A_0(r)}$$

where $A_0(r) = \pi r^2$ is the area of the Euclidean circle of radius r and $A(r)$ the circle on the surface.



Given the surface parametrically $(x(u,v), y(u,v), z(u,v))$ form and a curve $\gamma(t) = (u(t), v(t))$ on it. This forms a space curve given by

$$\Gamma(t) = (x(u(t), v(t)), y(u(t), v(t)), z(u(t), v(t)))$$

to compute its length we need to integrate $|\Gamma'(t)|$ which will be given by the chain rule as

$$\sqrt{(\frac{\partial x}{\partial u}u'(t) + \frac{\partial x}{\partial v}v'(t))^2 + (\frac{\partial y}{\partial u}u'(t) + \frac{\partial y}{\partial v}v'(t))^2 + (\frac{\partial z}{\partial u}u'(t) + \frac{\partial z}{\partial v}v'(t))^2}$$

This motivates us to make the following definition by setting

$$\begin{array}{ll} a = \frac{\partial x}{\partial u} & a' = \frac{\partial x}{\partial v} \\ b = \frac{\partial y}{\partial u} & b' = \frac{\partial y}{\partial v} \\ c = \frac{\partial z}{\partial u} & c' = \frac{\partial z}{\partial v} \end{array}$$

And then form

$$E(u,v) = a^2 + b^2 + c^2, F(u,v) = aa' + bb' + cc', G(u,v) = {a'}^2 + {b'}^2 + {c'}^2$$

to which we can associate the innerproduct

$$E(u,v)dudu' + F(u,v)(dudv' + du'dv) + G(u,v)dvdv'$$

This depends only on the 'interior' coordinates of the surface, and its length can thus be given by integrating $|\gamma'(t)|$ but now with respect to the above inner product on the surface.

Example: If we consider spherical coordinates, i.e. $x = \cos u \cos v, y = \sin u \cos v, z = \sin v$ we get the form $E = \cos^2 v, F = 0, G = 1$ which illustrates the fact that the length of a latitudinal circle at v will be scaled by $\cos v$.

Digression on Gauss-Bonnet

Gauss noted that for a geodesic triangle Δ with angles α, β, γ we have

$$\iint_\Delta K = \alpha + \beta + \gamma - \pi$$

which is a generalization of the fact that on a unit sphere, the area of a spherical triangle is equal to its angular excess (by noting that areas scale as R^2 and cirvature as R^{-2} this will hold for any sphere).

Now the angular excess is additive, which is easy to see thus there will be a function K' such that

$$\iint_\Delta K' = \alpha + \beta + \gamma - \pi$$

and it will simply be the limit of angular excess divided by area. If we can prove that this is equal to gaussian curvature we are done. It is true for spheres and the hyperbolic plane, and we would be done if we can well approximate surfaces with those locally.

Now given this we may make a formal calculation. Given a surface X we can triangulate it into geodesic triangles, and if n_0, n_1, n_2 denotes the number of vertices, edges and triangles respectively the euler characteristics $e(X)$ is given by

$$e(X) = n_0 - n_1 + n_2$$

Now add all the integrals $\iint_\Delta K$ and we get on the lefthand side $\iint_X K$ while on the right hand side we get $\sum_\alpha \alpha - \pi n_2$. The sum of all angles we can rearrange as to collect them vertice by vertice and than the right hand side becomes $2\pi n_0 - \pi n_2$. Now given n_2 triangles the number of edges will become $\frac{3}{2} n_2$ (each triangle give rise to three edges, but then every edge will be counted twice), thus $e(X) = n_0 - \frac{1}{2} n_2$ putting everything together we get Gauss-Bonnet

$$\iint_X K = 2\pi e(X)$$

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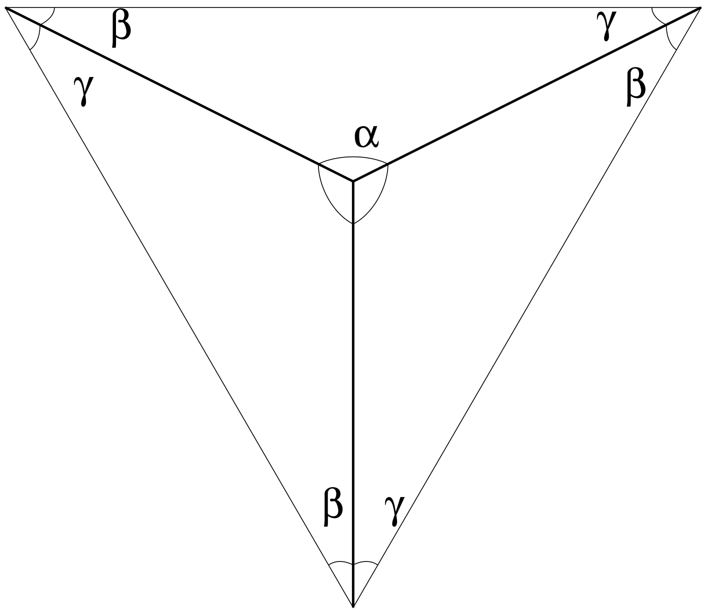
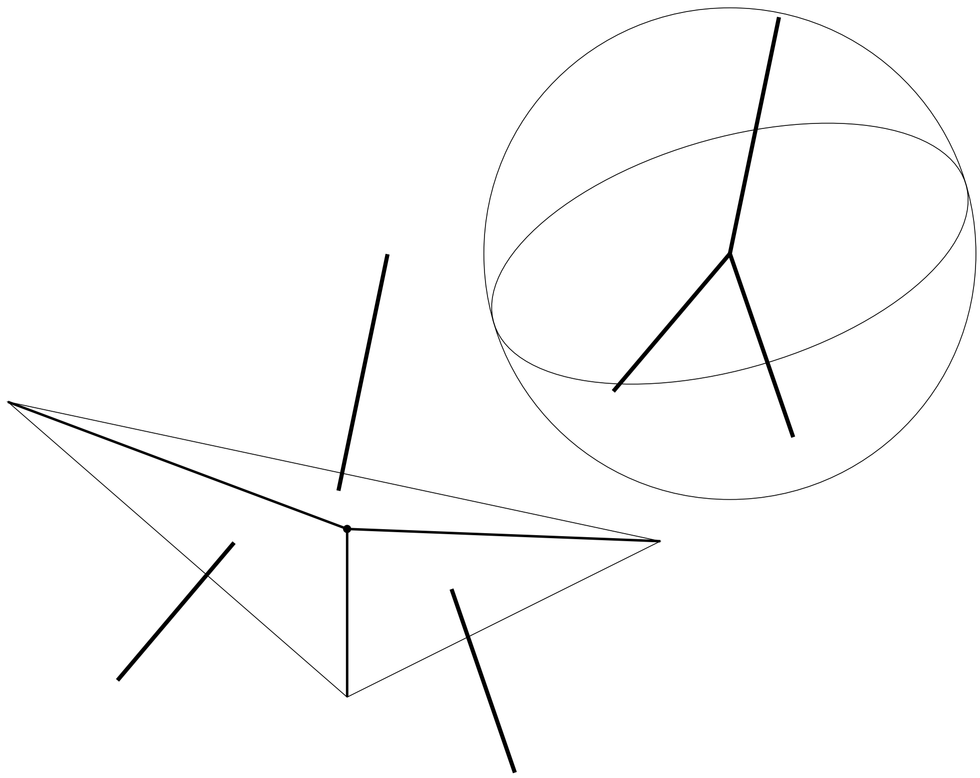


Given a polyheder we may look at a vertex. It will be associated to a number of angles α if $\sum_\alpha \alpha = 2\pi$ the polygons meeting at the corner will actually lie in a plane. If $\sum_\alpha \alpha < 2\pi$ we will have an ordinary convex corner, while if $\sum_\alpha \alpha > 2\pi$ we will have a saddle. It is convenient to introduce the factor $k = \frac{\sum_\alpha \alpha}{2\pi}$ Now given a circle of radius r centered at the vertex, its area will be $k\pi r^2$ and if we look at a polygon surrounding the vertix with n sides it will split up into n triangels with angles α, β, γ with $\alpha + \beta + \gamma = \pi$. The angular sum will be the sum of all the $\beta' s$ and $\gamma' s$ which will amount to $n\pi - \sum_\alpha \alpha = n\pi - 2k\pi = (n-2)\pi + 2(1-k)\pi$. We may think of all the curvature as concentrated at the corners given by $2(1-k)\pi$. Now if we add all the curvatures and rearrange the terms, we collect the $\alpha' s$ to corresponding polygons with n sides. The sum for each polygon will hence be $-(n-2)\pi$ and the total sum $-(2n_1 - 2n_0)\pi$ and hence the total $(2n_2 - 2n_1 + 2n_0) = 4\pi$, which is a discrete form of Gauss-Bonnet known to Descartes.



We may now relate excess angular sum with the Gauss curvature defined by the Gauss map. Consider the unit normals n_i to the faces f_i meeting at a fixed vertex. They will correspond to points p_i on the unit sphere, which can be seen as the dual of the configurations of the faces meeting at the vertix. Two adjacent points p_i, p_j are joined by a gedosic arc c_{ij} (meaning part of a great circle), the length of that arc corresponds to the angle the faces f_i, f_j meet along their edge e_{ij}, which incidentally is the angle at which the normals n_i, n_j meet. One may think of all the vectors v perpendicular to e_{ij} between n_i, n_j as normals along the edge. In this way we get a polygon on the unit sphere which will enclose a region P, corresponding so to speak to all the 'normals' at the vertex. In order to compute the area of P we need to compute all the angles formed by two adjacent arcs c_{ij}, c_{jk}. Now the edge e_{ij} is perpendicular to both n_i and n_j, thus we see that the sought after angle is related to the angle α_j between e_{ij} and e_{jk} (which is the angle of the face f_j at the vertex) or more precisely to their normals (in the plane spanned by the two edges) thus it will be given by $\pi - \alpha_j$. Adding everything up and computing the excess it will turn out to be exactly that of the angular excess of the polygon that encircles the vertex. Thus by approximating a surface with a polygon we can prove the theorem of Gauss above.

Hypergeometric Series, the Arithmetic-geometric mean and Elliptic Functions

Hypergeometric series

Gauss work on the hypergeometric function may not be the most exciting, yet it shows both his concern with rigor as well as his delight in formal manipulation of functions, where we once again may compare him to Euler.

There are various hypergeometric functions according to the number of parameters defined as follows

$$ {}_0F_1(a;z) = 1 + \sum_{n=1}^{\infty} \frac{1}{a(a+1)\dots(a+n-1)} \frac{z^n}{n!} $$

$$ {}_1F_1(a,b;c;z) = 1 + \sum_{n=1}^{\infty} \frac{a(a+1)\dots(a+n-1)}{b(b+1)\dots(b+n-1)} \frac{z^n}{n!} $$

$$ {}_2F_1(a,b;c;z) = 1 + \sum_{n=1}^{\infty} \frac{a(a+1)\dots(a+n-1)b(b+1)\dots(b+n-1)}{c(c+1)\dots(c+n-1)} \frac{z^n}{n!} $$

Many well-known functions are special cases of them, e.g.

$$ \begin{array}{cc} \log(1+z) = & z {}_2F_1(1,1;2;-z) \\ (1-z)^{-a} = & {}_2F_1(a,1:1:z) \\ \arcsin(z) = & z {}_2F_1(\frac{1}{2},\frac{1}{2};\frac{3}{2};z^2) \\ \cosh(z) = & {}_0F_1(\frac{1}{2};\frac{z^2}{4}) \\ \sinh(z) = & z {}_0F_1(\frac{3}{2};\frac{z^2}{4}) \\ \arctan(z) = & z {}_2F_1(\frac{1}{2},1;\frac{3}{2};-z^2) \end{array} $$

We can easily spot some formal properties such as

$$ 1 + a\frac{d {}_0F_1(a;z)}{dz} = -{}_0F_1(a+1;z) $$

or

$$ 1 + \frac{c}{ab}\frac{d {}_2F_1(a,b;c;z)}{dz} = -{}_2F_1(a+1,b+1;c+1;z) $$

They lend themselves naturally to continued fraction expansions. More specifically let $f_0, f_1 \dots$ be a sequence of analytic functions such that $f_{i-1} - f_i = k_i z f_{i+1}$ then $\frac{f_{i-1}}{f_i} = 1 + k_i z \frac{f_{i+1}}{f_i}$ i.e. $\frac{f_i}{f_{i-1}} = \frac{1}{1+k_i z \frac{f_{i+1}}{f_i}}$. Now setting $g_i = \frac{f_{i-1}}{f_i}$ we can thus set $g_i = \frac{1}{1+k_i z g_{i+1}}$ we get

$$ g_1 = \frac{f_1}{f_0} = \frac{1}{1+k_1 z g_2} = \frac{1}{1+\frac{k_1 z}{1+k_2 z g_3}} = \frac{1}{1+\frac{k_1 z}{1+\frac{k_2 z}{1+k_3 z g_4}}} = \dots $$

leading to the continued fraction

$$\frac{f_1}{f_0} = \cfrac{1}{1+\cfrac{k_1 z}{1+\cfrac{k_2 z}{1+\cfrac{k_3 z}{1+\ldots}}}}$$

For the simplest hypergeometric series we may start with the identity

$${}_0F_1(a-1;z) -_0 F_1(a;z) = \frac{z}{a(a-1)}{}_0 F_1(a+1;z)$$

and hence take $f_i =_0 F_1(a+i;z), k_i = \frac{1}{(a+i)(a+i-1)}$ leading to

$$\frac{{}_0F_1(a+1;z)}{a_0F_1(a;z)} = \cfrac{1}{a+\cfrac{z}{(a+1)+\cfrac{z}{(a+2)+\cfrac{z}{(a+3)+\ldots}}}}$$

With more elaborate identities on may establish

$$\frac{{}_2F_1(a+1,b;c+1;z)}{c_2F_1(a,b;c,z)} = \cfrac{1}{c+\cfrac{(a-c)bz}{(c+1)+\cfrac{(b-c-1)(a+1)z}{(c+2)+\cfrac{(a-c-1)(b+1)z}{(c+3)+\cfrac{(b-c-2)(a+2)z}{(c+4)+\ldots}}}}}$$

Using ${}_2F_1(0,b;c;z) = 1$ setting $a = 0$ and replacing c with $c+1$ we obtain the simpler version

$${}_2F_1(1,b;c;z) = \cfrac{1}{1+\cfrac{-bz}{c+\cfrac{(b-c)z}{(c+1)+\cfrac{-c(b+1)z}{(c+2)+\cfrac{2(b-c-1)z}{(c+3)+\cfrac{-(c+1)(b+2)z}{(c+4)+\ldots}}}}}}$$

As results we get[31]

$$\tanh(z) = \cfrac{z}{1+\cfrac{z^2}{3+\cfrac{z^2}{5+\cfrac{z^2}{7+\ldots}}}}$$

$$\tan(z) = \cfrac{z}{1-\cfrac{z^2}{3-\cfrac{z^2}{5-\cfrac{z^2}{7-\ldots}}}}$$

The Arithmetic-Geometric Mean

Another thing Gauss played with was the arithmetic-geometric mean (agM) from now on denoted by $M(a,b)$ of two numbers a, b. Assume that $0 < a < b$ and form the geometric mean $a_1 = \sqrt{ab}$ and the arithmetic mean $b_1 = \frac{a+b}{2}$ getting two numbers a_1, b_1 nestled in the former i.e. $a < a_1 < b_1 < b$. Proceed inductively producing a_n, b_n. Those number will converge to the agM $M(a,b)$, in fact in view of

$$b_1 - a_1 = \frac{b_1^2 - a_1^2}{a_1 + b_1} = \frac{(a-b)^2}{2(a_1+b_1)}$$

and thus inductively

$$b_{n+1} - a_{n+1} = \frac{b_{n+1}^2 - a_{n+1}^2}{a_{n+1} + b_{n+1}} = \frac{(a_n - b_n)^2}{2(a_{n+1} + b_{n+1})}$$

we see that the convergence is very fast. More precisely an error of ϵ at one stage is reduced to an error of $\frac{\epsilon^2}{4M(a,b)}$ at the next. As an example one may try to expand $M(1+x, 1-x)$ in a power series. As the function is symmetric with respect to its two variables it must be even ($x \mapsto -x$ makes no difference) and thus only involving even powers, i.e. being a power series in x^2. We also note that the initial error is bounded by $2x$ at the second stage it will be bounded by x^2 then by $\frac{1}{4}x^4$ and then by $\frac{1}{64}x^8$ etc. If we with obvious notation look at the power series $(1+x)_n, (1-x)_n$ we see that they will coincide with the limit one at terms up to 2^n (or some such number)[32]. We can work out by hand the first approximations using the formula [33]

$$\sqrt{1-a} = 1 - \frac{1}{2}a - \frac{1}{8}a^2 - \frac{1}{16}a^3 - \frac{5}{128}a^4 - \frac{7}{512}a^5 - \frac{21}{1024}a^6 - \frac{33}{2048}a^7 - \frac{429}{32768}a^8 + \dots$$

in which various power series will will substituted for a

n	$(1+x)_n$	$(1-x)_n$
0	$1+x$	$1-x$
1	1	$1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \frac{5}{128}x^8 + \dots$
2	$1 - \frac{1}{4}x^2 - \frac{1}{16}x^4 - \frac{1}{32}x^6 - \frac{5}{256}x^8 + \dots$	$1 - \frac{1}{4}x^2 - \frac{3}{32}x^4 - \frac{7}{128}x^6 - \frac{77}{2048}x^8 + \dots$
3	$1 - \frac{1}{4}x^2 - \frac{5}{64}x^4 - \frac{11}{256}x^6 - \frac{117}{4096}x^8 + \dots$	$1 - \frac{1}{4}x^2 - \frac{5}{64}x^4 - \frac{3}{64}x^6 - \frac{243}{8192}x^8$

A rather tedious procedure from which we see that $M(1+x, 1-x) = 1 - \frac{1}{4}x^2 - \frac{5}{64}x^4 + \dots$. One may easily mechanize the procedure and write some simple C-code and end up with

$1 - \frac{1}{4}x^2 - \frac{1}{16}x^4 - \frac{1}{32}x^6 - \frac{5}{256}x^8 - \frac{7}{512}x^{10} - \frac{21}{2048}x^{12} - \frac{33}{4096}x^{14} - \frac{429}{65536}x^{16}$

$1 - \frac{1}{4}x^2 - \frac{3}{32}x^4 - \frac{7}{128}x^6 - \frac{77}{2048}x^8 - \frac{231}{8192}x^{10} - \frac{1463}{65536}x^{12} - \frac{4807}{262144}x^{14} - \frac{129789}{8388608}x^{16}$

$$1-\tfrac{1}{4}x^2-\tfrac{5}{64}x^4-\tfrac{11}{256}x^6-\tfrac{117}{4096}x^8-\tfrac{343}{16384}x^{10}-\tfrac{2135}{131072}x^{12}-\tfrac{6919}{524288}x^{14}-\tfrac{184701}{16777216}x^{16}$$
$$1-\tfrac{1}{4}x^2-\tfrac{5}{64}x^4-\tfrac{11}{256}x^6-\tfrac{235}{8192}x^8-\tfrac{693}{32768}x^{10}-\tfrac{8683}{524288}x^{12}-\tfrac{28325}{2097152}x^{14}-\tfrac{1522109}{134217728}x^{16}$$

$$1-\tfrac{1}{4}x^2-\tfrac{5}{64}x^4-\tfrac{11}{256}x^6-\tfrac{469}{16384}x^8-\tfrac{1379}{65536}x^{10}-\tfrac{17223}{1048576}x^{12}-\tfrac{56001}{4194304}x^{14}-\tfrac{2999717}{268435456}x^{16}$$
$$1-\tfrac{1}{4}x^2-\tfrac{5}{64}x^4-\tfrac{11}{256}x^6-\tfrac{469}{16384}x^8-\tfrac{1379}{65536}x^{10}-\tfrac{17223}{1048576}x^{12}-\tfrac{56001}{4194304}x^{14}-\tfrac{5999435}{536870912}x^{16}$$

From this we note some computational mistakes from the hand calculation (Gauss of course did all his calculation by hand, many of them in his head, and only rarely making mistakes, even when they were very extensive and involved) but more interestingly we note (as expected) that the error of the first pair is x^4 of the second pair x^8 and the last pair x^{16} and we can write down

$$1-\frac{1}{4}x^2-\frac{5}{64}x^4-\frac{11}{256}x^6-\frac{469}{16384}x^8-\frac{1379}{65536}x^{10}-\frac{17223}{1048576}x^{12}-\frac{56001}{4194304}x^{14}+\dots$$

The point is not to compute the power series as a computational tool, as this is not what power series are intended to be used as[34], that is in this case effected much more rapidly in the direct way, but to see some pattern and hence to get an independent characterization of it. For that purpose Gauss used another approach. Namely we can write

$$M(1+\frac{2t}{1+t^2},1-\frac{2t}{1+t^2}=\frac{1}{1+t^2}M((1+t)^2,(1-t)^2)=\frac{1}{1+t^2}M(1+t^2,1-t^2)$$

From this Gauss considers the expansion of

$$\frac{1}{M(1+x,1-x)}=1+Ax^2+Bx^4+Cx^6+\dots$$

Setting $x=\frac{2t}{1+t^2}$ we get from the above the identity

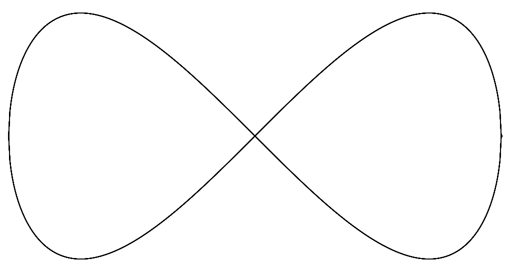
$$1+A(\frac{2t}{1+t^2})^2+B(\frac{2t}{1+t^2})^4+C(\frac{2t}{1+t^2})^2+\dots=(1+t^2)(1+At^4+Bt^8+Ct^{12}+\dots$$

Now we need to identify coefficients and solve for the $A, B, C \dots$. We can readily work out $A=\frac{1}{4}$ and $B=\frac{9}{64}$. Through a *tour de force* Gauss is able to simplify those equations to $4A=1, 16B=9A, 36C=25B\dots$ thus finding a closed form for the expansion of the inverted function

$$\frac{1}{M(1+x,1-x)}=\sum_{n=0}^{\infty}\left(\frac{1\cdot 3\cdot 5\cdot 7\cdot\dots\cdot(2n-1)}{2^n n!}\right)^2 x^{2n}$$

The Lemniscate and Elliptic integrals and functions

Let us now turn to the lemniscate which was introduced by the Bernoulli brothers in the 18th century.



The lemniscate is the locus of points whose product (not the sum) of distances to two given points is constant.

> In fact by varying the value of the product we get a whole family of curves, known as Cassini's ovals. They are actually the level curves of a quartic polynomial, and the leminiscate proper is the one which passes through the saddle located half-way between the two reference points, where it will have a node. If those are given by $(\pm a, 0)$ the quartic will be given by
>
> $$(x^2 + y^2 + a^2)^2 - (4a^2x^2 + a^4) = 0$$
>
> and in the sequel we will find it convenient to set $2a^2 = 1$ and the lemniscate will cross the x-axis at $\pm 1, 0$ given by $x^2(x^2 - 1) = 0$ with a double zero at the origin due to the singularity.

Now it is natural to use the radius vector $r = \sqrt{x^2 + y^2}$ to parametrize the lemniscate. Let us write the equation on the form

$$(x^2 + y^2)^2 + (x^2 + y^2) - 2x^2 = 0$$

or $x^2 = \frac{1}{2}(r^2 + r^4)$ and hence $y^2 = \frac{1}{2}(r^2 - r^4)$. We get $2x\dot{x} = (r + 2r^3), 2y\dot{y} = (r - 2r^3)$ thus

$$\dot{x}^2 + \dot{y}^2 = \frac{(r + 2r^3)^2}{r^2 + r^4} + \frac{(r - 2r^3)^2}{r^2 - r^4} = \frac{(1 + 2r^2)^2}{1 + r^2} + \frac{(1 - 2r^2)^2}{1 - r^2} = \frac{1}{1 - r^4}$$

Hence the arc length of the lemniscate is given by the integral $\int \frac{1}{\sqrt{1-r^4}}$ in particular a quarter of the length of the lemniscate is given by the integral

$$\int_0^1 \frac{dr}{\sqrt{1-r^4}}$$

This integral was well-known during the 18th century (in fact it appeared in a paper by Jacob Bernoulli already in 1691) and had been studied by Euler and Lagrange among others. Gauss even had a special notation $\tilde{\omega}$ for its value setting

$$\tilde{\omega} = 2\int_0^1 \frac{dr}{\sqrt{1-r^4}}$$

computing the value $\frac{\pi}{\tilde{w}}$ numerically to eleven digits he noticed that it coincided with $M(1,\sqrt{2}) = 1.1981402347355922074...$ He then set out to prove it. One method was to show that

$$M(a,b)\cdot\int_0^{\frac{\pi}{2}}(a^2\cos^2\phi + b^2\sin^2\phi)^{-\frac{1}{2}} = \frac{\pi}{2}$$

the key step being that $I(a,b) = I(a_1,b_1)$ where I is the above integral and a_1, b_1 occur after the first step in the agM process, noting that $(a_n^2\cos^2\phi + b_n^2\sin^2\phi)^{-\frac{1}{2}}$ converges uniformly to $\frac{1}{M(a,b)^{\frac{1}{2}}}$. He does that by introducing the variable substitution

$$\sin\phi = \frac{2a\sin\phi'}{a+b+(a-b)\sin^2\phi'}$$

which will lead to

$$(a^2\cos^2\phi + b^2\sin^2\phi)^{-\frac{1}{2}}d\phi = (a_1^2\cos^2\phi' + b_1^2\sin^2\phi')^{-\frac{1}{2}}d\phi'$$

and he leaves out the details[35].

Now at the time well-known elliptic integral

$$F(k,\frac{\pi}{2}) = \int_0^{\frac{\pi}{2}}(1-k^2\sin^2\phi)^{-\frac{1}{2}}d\phi$$

is related to $I(a,b)$ as follows. Set $k = \frac{a-b}{a+b}$ then we have

$$I(a,b) = \frac{1}{a}F(\frac{2\sqrt{k}}{1+k},\frac{\pi}{2})$$

Furthermore by setting $r = \cos\phi$ we get

$$\int_0^1 \frac{dr}{\sqrt{1-r^4}} = \int_0^{\frac{\pi}{2}}(2\cos^2\phi = \sin^2\phi)^{-\frac{1}{2}}d\phi = I(\sqrt{2},1)$$

Now impressive as those calculations and manipulations may be, the most significant thing he did was to invert elliptic integrals, in fact the lemniscate integral above, and getting double periodic functions which he denoted by sinlemn (]sln) and coslem (cl) in analogy with the trigonometric functions. In so doing he anticipated Abel and Jacobi. More precisely

$$\mathrm{sl}(\int_0^x (1-z^4)^{-\frac{1}{2}}dz) = x$$

and

$$\mathrm{cl}(\frac{\tilde{\omega}}{2} - \int_0^x (1-z^4)^{-\frac{1}{2}}dz) = x$$

from which will follow identities such as

$$\mathrm{sl}^2\phi + \mathrm{cl}^2\phi = \mathrm{sl}^2\phi\mathrm{cl}^2\phi = 1$$

and

$$\mathrm{sl}(\phi+\phi') = \frac{\mathrm{sl}\phi\mathrm{cl}\phi' + \mathrm{sl}\phi'\mathrm{cl}\phi}{1-\mathrm{sl}\phi\mathrm{sl}\phi'\mathrm{cl}\phi\mathrm{cl}\phi'}$$

which are actually identities that go back to Euler but in different garb. Furthermore he anticipated Jacobi by designing theta functions, in which they could be expressed as quotients. The periods he computed as $2\pi G, 2\pi iG$ (Note that the lattice generated is up to scale the same as the one of Gaussian integers) where

$$G = \frac{2}{\pi}\int_0^1 \frac{dt}{\sqrt{1-t^4}} = \frac{\tilde{\omega}}{\pi}$$

Magnetism, Gauss Theorem

Gauss hade already met Wilhelm Weber (184-91) at a meeting in Berlin in 1828 (Gauss attended few meetings) and been impressed by him. In 1831 Weber became a professor of physics at Göttingen and a co-operation started. Gauss had always been interested in physics, not only astronomy, but until then his work had been theoretical, Most importantly in a short paper he had built on the ideas of Maupertuis and d'Alembert on least actions and developed a principle of least constrain in which he sought to unify mechanics. With Weber experimental work began, mostly concerned with magnetism, including (on the suggestion of the explorer Alexander Humboldt, who in vain tried to entice Gauss to come to Berlin) the mapping of the magnetic field of the Earth, for which purpose they even founded a journal. The significance of his work in physics has been recognized as the unit G of the strength of magnetic fields in the now superseeded cgs system[36]. Mathematically Gauss was a pioneer introducing potentials and in this context the divergence theorem or Gauss theorem, concerns the integral of the flux of a vector field $\mathbf{F}$ across a surface S plays a central role, as all students of mathematics are aware of. Explicitly

$$\iiint_V (\nabla\cdot\mathbf{F})dV = \iint_S (\mathbf{F}\cdot\mathbf{n})dS$$

where $\nabla \cdot \mathbf{F} = \text{div}\mathbf{F} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}$ of particular interest being the case when the divergence vanishes outside the singularities of the field $\mathbf{F}$. In other words when it is the gradient field of a harmonic function, which occurs naturally when gravity or electrical attraction is concerned, because of the inverse square law.

Gaussian distributions and Least Squares

Gauss was very interested in observational astronomy and hence concerned with error of observations. At every observation there will be mistakes, some random some principal. It is the business of theory to rule out the latter, but as to the random fluctations there is nothing to be done except to expect that they will somehow cancel out in the long run. Thus we need to make many observations to determine a quantity, say $a_1, \dots a_n$ what is the 'true' value x of the observed entity, or better still an x which differ from the values a_n as little as possible? There is no canonical answer to this question but a natural norm in this case is the one given by the sum of squares. In other words we want to find x as to make $\sum_n (x - a_n)^2$ as small as possible. Differentiating we find the condition $0 = \sum_n (x - a_n) = nx - \sum_n a_n$ i.e. $x = \frac{1}{n} \sim_n a_n$. In terms of linear algebra we can consider the vectors $A = (a_1, \dots a_n), I = (1, \dots 1)$ and seeking to minimize $< A - xI \cdot A - xI >$ under the standard inner product. Geometrically this amounts to finding the projection of A onto the subspace spanned by I. This can be generalized. Say that we are given a number of dots (a_n, b_n) in the plane and want to find the equation of a line that 'best fits'. Setting the equation of the line as $Xx + Yy + Z = 0$ where X, Y, Z to be determined we naturally want to minimize the sum of squares $\sum_n (Xa_n + Yb_n + Z)^2$ or setting $A = (a_n), B = (b_n)$ the squared norm $< AX + BY + ZI \cdot AX + BY + ZI >$. An obvious and uninteresting solution is $X = 0, Y = 0, Z = 0$, uninteresting as it does not correspond to a line. It is then natural to normalize (X, Y, Z) such that $X^2 + Y^2 + Z^2 = 1$. Using the technique of Lagrange multipliers we need to find out when the two gradients are parallel. This leads to the system of equations

$$\begin{array}{rl} < A \cdot A > X + < A \cdot B > Y + < A \cdot I > Z = & \lambda X \\ < A \cdot B > X + < B \cdot B > Y + < B \cdot I > Z = & \lambda Y \\ < A \cdot I > X + < I \cdot B > Y + < I \cdot I > Z = & \lambda Z \end{array}$$

This can be reformulated as finding the eigenvectors to the linear transformation T whose symmetric matrix is given by

$$\begin{pmatrix} < A \cdot A > & < A \cdot B > & < A \cdot I > \\ < A \cdot B > & < B \cdot B > & < B \cdot I > \\ < A \cdot I > & < I \cdot B > & < I \cdot I > \end{pmatrix}$$

It corresponds to the linear map $v \mapsto < A \cdot v > A + < B \cdot v > B + < C \cdot v > C$. A more natural clarification will appear below in the digression on quadratic forms.

Another normalization may be done by setting $y = -1$ corresponding to finding the best linear relation $y = Xx + Z$ giving the values b_n when the a_n

are plugged in. In this case we look at the two partials with respect to X and Z setting them zero

$$< A \cdot A > X + < A \cdot I > Z = \quad < A \cdot B >$$
$$< A \cdot I > X + < I \cdot I > Z = \quad < I \cdot B >$$

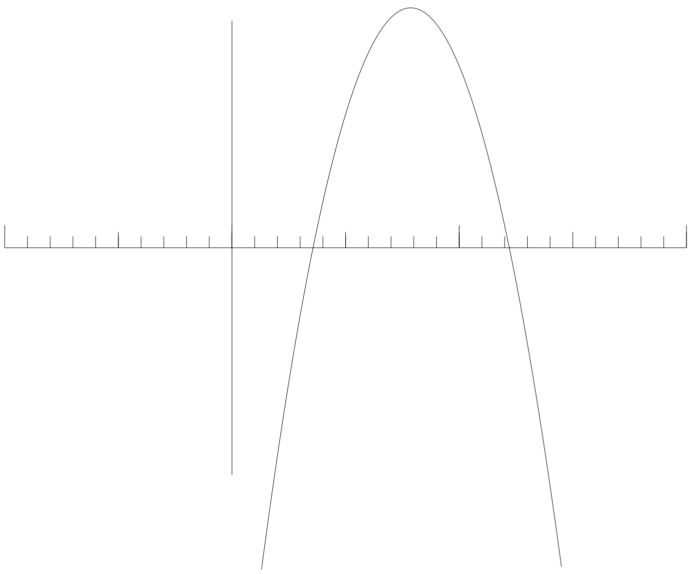
Let us look at the case $(1,1),(3,4),(2,5)$ with $A=(1,3,2), B=(1,4,5), I=(1,1,1)$ We get the matrix

$$\begin{pmatrix} 14 & 23 & 6 \\ 23 & 42 & 10 \\ 6 & 10 & 3 \end{pmatrix}$$

and a corresponding cubic characteristic equation

$$\lambda^3 - 59\lambda^2 + 91\lambda - 25 = 0$$

You expect three real roots corresponding to the axi of an ellipsoid (in case we have a rotational one you expect one eigenvalue to be double and a corresponding 2-dimensional vector space). In our case we get two small roots $0.357, 1.22$ and a big root 57.423



We find the normalized eigenvectors in the three cases and the corresponding sum of squares

eigenvalue	eigenvector	sum of squares
0.357	$(0.452, -0.036, -0.891)$	0.357
1.22	$(0.752, -0.523, 0.402)$	1.22
57.423	$(-0.48, -0.852, -0.209)$	57.423

Note the curious fact that the sum of the squares equals the corresponding eigenvalues. It could hardly be a coincidence?

As to the second method we only need to solve the system of equations

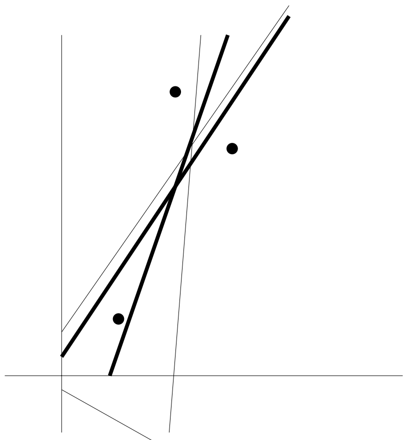
$$\begin{aligned} 14X + 6Z &= 23 \\ 6X + 3Z &= 10 \end{aligned}$$

with the solution $X = \frac{3}{2}, Z = \frac{1}{3}$.

We could also have done it expressing Y as a function of X then the relevant system would have been

$$\begin{aligned} 42Y + 10Z &= 23 \\ 10Y + 3Z &= 6 \end{aligned}$$

with the solution $Y = \frac{9}{26}, Z = \frac{11}{13}$. Those two are plotted below in fat, along with the three stationary lines from the first method.



We note that the line that has the best fit in the latter case is not the one close to the fat one.

The first method is mathematically more satisfying or at least elegant, but as we have noted computationally much more complicated. The method can be vastly generalized, a circle can be given by three linear parameters by $x^2 + y^2 + Xx + Yy + Z = 0$ while for a general quadric we need six linear parameters that need to be normalized. More precisely: If the points are given by (a_n, b_n) there will be three quadric monomials $a_n^2, a_n b_n, b_n^2$ two linear ones a_n, b_n and one contant one 1. Thus we will consider $\sum(x_1A_1 + x_2A_2 + x_3A_3 + x_4B_1 + x_5B_2 + x_6)^2$ where A_i, B_i etc are vectors determined by observational data. In general we will consider $\sum_i(\sum_k x_kA_k)^2$ where the vectors A_k are concocted in various ways from the data.

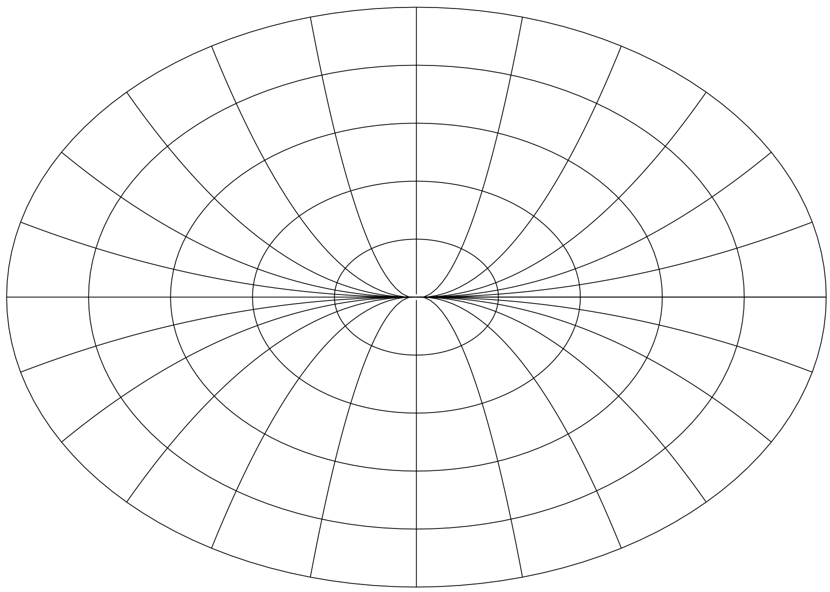
To consider a sum of squares

$$\Sigma = \sum_k (x_1 A_k + x_2 B_k + \dots x_n \Omega_n)^2$$

as above is to consider a homogenous quadric in the variables $x_1, x_2 \dots x_n$. Given any homogeneous polynomial $F(x_1, x_2 \dots x_n)$ of degree d (i.e. $F(tx_1, tx_2 \dots tx_n) = t^d F(x_1, x_2 \dots x_n)$) we can form $F_0 = \sum_i x_i \frac{\partial F}{\partial x_i}$ and it turns out that $F_0 = dF$ a relation named after Euler. It is easy to verify as the relation is obviously linear and it is enough to look at monomials. In our case we have a matrix M whose rows are half of the partials. If we have an eigenvector $\Xi = (\xi_1, \xi_2 \dots xi_n)$ on the unit sphere with eigenvalue λ for the matrix we have

$$\Sigma(Xi) = \sum_i \xi_i(\lambda x_i) = \lambda \sum_i x_i^2 = \lambda$$

and we have verified the statement we came across numerically.



To understand it properly we note that any quadric Q gives rise to a gradient vector field on the vector space V (of dimension n)[37]. As the tangents have canonical identifications by the underlying linear space itself this gives rise to a map T from V to V. Furthermore the potential being quadratic means that T is linear (and its matrix is $2M$ as above). The level surfaces of Q are ellipsoids and as we put a ballon at the origin and blow it up it will touch the ellipsoid n-times corresponding to its axi. The first it encounter will be the minimal one, and the last the maximal one, which will correspond to the minimum and maximum respectively of the distance to the origin, for the remaining ones we have saddles.

Finally we can clarify everything with some more abstract notation. What we have been doing is to create from a given inner product $< * \cdot * >$ a quadratic form $< Lv \cdot Lv >$ on V where L is a linear map. Now every quadratic form can be given as $< Av, v >$ where A is a linear map on V (corresponding to the one we represented by T above and a symmetric matrix M). We can represent A as $A = L^*L$ which becomes symmetric. Now if v with $< v \cdot v >= 1$ is a eigenvector with respect to A with eigenvalue λ we get

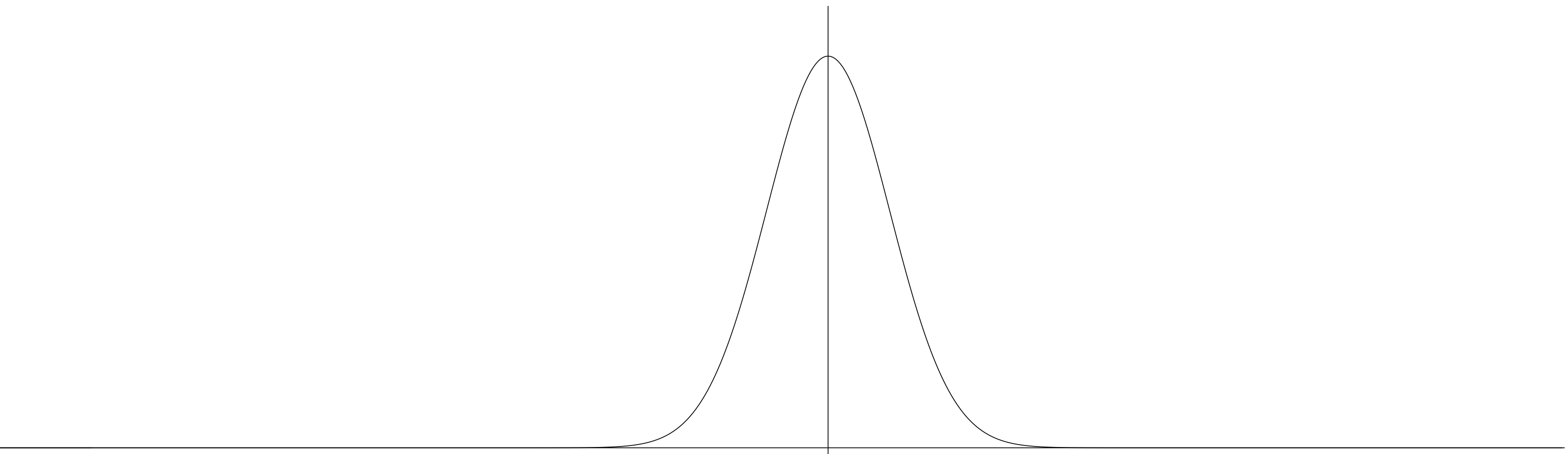
$$< Lv \cdot Lv >=< Av \cdot v >= \lambda < v, \cdot v >= \lambda$$

We also note that eigenvectors v, w corresponding to different eigenvalues λ, μ will be orthogonal as

$$\lambda < v \cdot w >=< Av \cdot w >=< v \cdot Aw >= \mu < v \cdot w >$$

Hence the distinct axi of the ellipsoid are orthogonal.

Gaussian distributions



The normal distribution is given by

$$\frac{1}{\sqrt{2\sigma^2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where μ is the expected average and σ^2 is the variance (σ is referred to as the standard deviation). The crucial function is $\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ where the coefficient is chosen to make the integral $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = 1$[38]. By scaling with $\frac{1}{\sqrt{2}\sigma}$ and translating with μ we get the general case.

Say that we want to measure a certain quantity μ then there will be some observational error. Those errors are the compound of several small errors, sometimes reinforcing each other, sometime canceling each other out. If we make many observations and take their average we expect that the result will converge to the true value. We will make this more precise below.

As an example assume that we are tossing a fair coin, meaning that we expect each face to turn up equally often. Thus if we make n tosses and observe the frequency of each face we expect that the actual observation will differ slightly from the true value because of error. If we consider all the 2^n possible sequences of head and tails and keep track of how many of those will result in k heads say we get the binomial distribution with $\binom{n}{k}$ such sequences. Now normalize it by considering k ranging between $-n/2$ and $n/2$ by defining $\phi_n(k) = \binom{n}{k+n/2}/2^n$. The integral of this step function will be 1 but as $n \to \infty$ we get that $\phi_n(x) \to 0$ spread along an interval of length n. If we scale this interval by a factor $\frac{2}{\sqrt{n}}$ thus going from $-\sqrt{n}$ to $\sqrt{n}$ then we have to scale the function by $\sqrt{n}/2$ to keep the area equal to one. Thus we are looking at $\psi_n(x) = \lambda_n \phi_n(\lambda_n x)$ with $\lambda_n = \sqrt{n}/2$. Using Stirling's formula

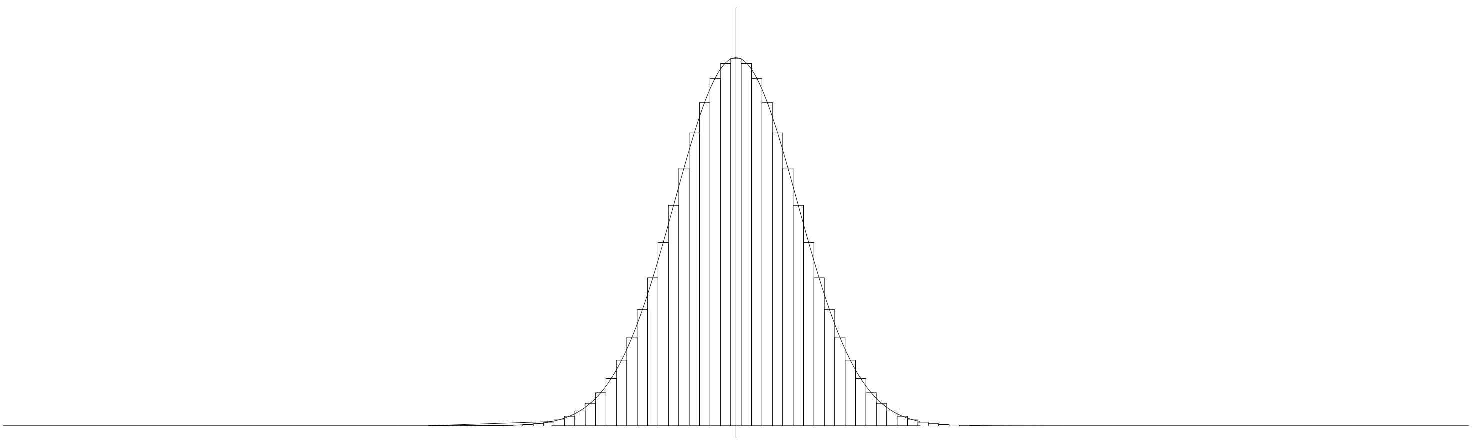
$$n! \sim \sqrt{2\pi n} n^n e^{-n}$$

we can write down

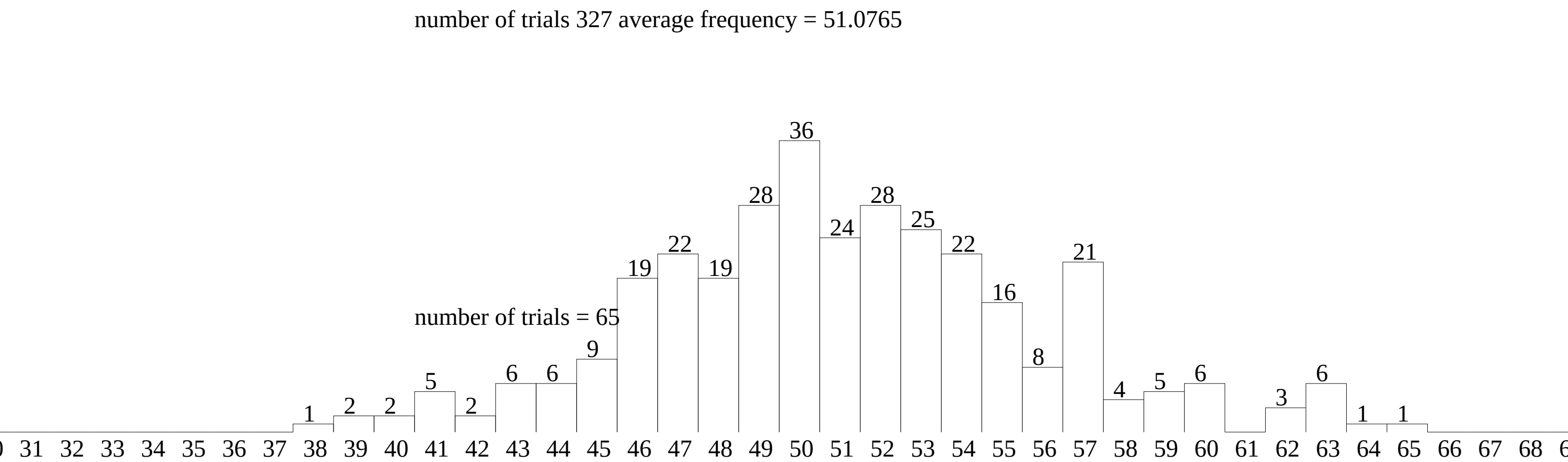
$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2} \sim \frac{\sqrt{4\pi n} \cdot 2^{2n} n^{2n} e^{-2n}}{(2\pi n) n^{2n} e^{-2n}} = 2^{2n} \frac{2\sqrt{\pi}}{2\pi\sqrt{n}}$$

Thus $\psi_{2n}(0) = \frac{1}{\sqrt{2\pi}}$ and ψ_n will converge to $\psi = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ [39]

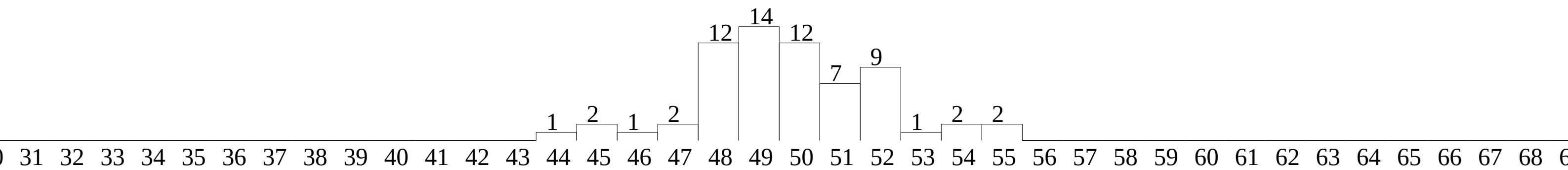
In the picture below we see an approximation to the normal distribution by the binomial.



We can now perform an experiment tossing a coin one hundred times and noting the frequencies of heads. Redoing it some three hundred times we end up with the following distribution of observations.



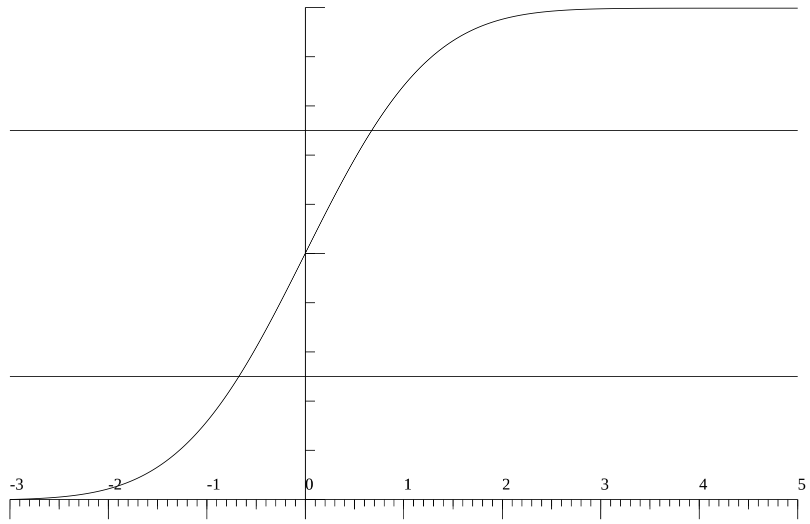
If we instead had tossed it five hundred times and noting the frequencies of heads, we would instead have got the following distribution



We see that it is more concentrated. This shows that taking the average value of a number of observation increases the accuracy. We can be even more precise.

The value of the integral $\frac{1}{\sqrt{2\pi}}\int_x^y e^{-\frac{1}{2}t^2}dt$ gives the probability that an observation will land in the interval $[x,y]$. If we are tossing N coins the interval $[x,y]$ corresponds that the number of heads lies in the interval $[\frac{N}{2}+x\sqrt{N}/2, \frac{N}{2}+y\sqrt{N}/2]$

We can look at the cumulative function $\frac{1}{\sqrt{2\pi}}\int_\infty^x e^{-\frac{1}{2}t^2}dt$



If we look at the interval given by the inverse of $[\frac{1}{4}, \frac{3}{4}]$ in which half the values occur, we see that this is given by $[-0.67\ldots, 0.67\ldots]$. Thus with $N = 100$ throws, we expect about half of the throws to occur in the interval $[50 - 0.67 \cdot 5, 50 + 0.67 \cdot 5]$ and indeed between 46 and 54 we have about 180

More generally if we make two attempts of N trials they can be thought of as one attempt of $2N$ trials which can be thought of as the sum of the two trials. Both, however, give rise to Gaussian distributions. Thus if we have an error corresponding to x from the average[40] it means in the first case a deviation of $x = \sqrt{N}/2$ in the second case $x = \sqrt{2N}2$ thus a spread $\sqrt{2}$ as wide in as in the first case. If we think in terms of random walks, the maximal deviation from the equilibrium grows with time, as is intuitively clear, and in fact by the square root of the time (or number of trials). However, if we think of the relative deviation dividing with $N/2$ and N respectively we get errors of $\frac{1}{\sqrt{N}}$ and $\frac{1}{\sqrt{2}}\frac{1}{\sqrt{N}}$ respectively and the relative error is decreased by a factor $\frac{1}{\sqrt{2}}$ instead. More generally the relative errors decrease like $\frac{1}{\sqrt{N}}$ when you make N trials, which can be thought of as taking the average of N trials. An accuracy of ϵ for one observation (say meaning 50 % chance of an error less than ϵ) will now be reduced to an error of $\frac{1}{\sqrt{N}}\epsilon$. We may think of it as scaling the variable x with $\sigma = \frac{1}{\sqrt{N}}$ giving a more pointed distribution less spread out. Its variance, as defined above, will be σ^2.

Notes

[1]Why not King of Mathematics? There is no separate English word for the German *Furst* or the Swedish *furste* which denotes the monarch of a small kingdom, or principality (*furstendöme*), of which Germany was a mosaic.

[2]The series is usually presented as the sum of the first hundred numbers, most likely it was more complicated and that its arithmetic nature was not immediate.

[3]Both Euler and Gauss made no distinction between pure and applieed mathematics, everything was food for thought, and in the case of Gauss he is said to have quoted King Lear to the effect of *Thou, nature, art my goddess; to thy laws my services are bound.*

[4]He also left descendants, of whom Eugene and Albert Fawcett lived until the early 1950's, but neither seem to have left any issue. The longest living grandchild of Gauss was Jospeh Henry (1855-1956) son of Charles Gauss (1813-79) and born in the Missouri. His daughter Janet Lee (1912-2012) must be the longest surviving great grandchild of Gauss. There are still great great grandchildren about, all Americans.

[5]Less spectacular students were Moebius, who nevertheless is a name that pops up repeatedly and may to most mathematicians be more well-known than Dirichlet. Even more than that of Riemann among the general population.

[6]The case $p = 257$ was worked out in a dissertation in 1831 involving a sequence of seven quadratic extensions by Friedrich Julius Richelot (1808-75) a student of Jacobi, whom he later succeeded at Königsberg. The construction involved 194 printed pages. Incidentally Richelot was the father-in-law of the well-known 19th century physicist Kirchoff. Attempts at the case of $p = 65537$ involving fifteen quadratic extensions were undertaken by Johann Gustav Hermes (1846-1912) who devoted ten years to the task. The results of his efforts, completed in 1889, remain in some large chest specially designed and made for that purpose at the Göttingen math department. A shorter report on it was published in 1894, sponsored by Felix Klein.

[7] If the polynomial is reducible, it splits up into two linear factors and have two rational (unrelated) roots. To talk about numbers may be a trifle misleading as it is not strictly necessary, one may look at the ring $\mathbb{Q}[x]/(x^2 - px + q)$ where we will have the rule $x^2 = px - q$, thus in practice (see below) any element can be written as $ax + b$ with the above rule.

[8]It turns out that $\eta \mapsto \eta'$ is both linear and multiplicative, in fact the one non-trivial automorphism of a quadratic extension.

[9]There is some confusion what is a discriminant. If the linear term is even one can factor out the factor 4 in the discriminant, thus $4b^2 - 4ac$ is reduced to $b^2 - ac$ as Gauss does in his definition. The confusion is irritating, but not serious, as a discriminant is mostly interesting mod squares anyway.

[10]Gauss considers integral forms of type $ax^2 + 2bxy + cy^2$ which corresponds to lattices for which the restriction of the inner product is integral, which is a strong condition. A weaker, and perhaps more natural condition is that the restriction of the quadratic form is integral.

[11]Given two vectors ω_1, ω_2 the area Δ of the parallelogram spanned by them is given by $\sin\theta|\omega_1||\omega_2|$ from which follows that

$$\Delta^2 = \sin^2\theta|\omega_1|^2|\omega_2|^2 = |\omega_1|^2|\omega_2|^2 - < \omega_1 \cdot \omega_2 >^2$$

as

$$< \omega_1 \cdot \omega_2 >= \frac{1}{2}(\omega_1\bar{\omega}_2 + \omega_2\bar{\omega}_1)$$

we get

$$< \omega_1 \cdot \omega_2 >^2= \frac{1}{4}(\omega_1^2\bar{\omega}_2^2 + \omega_2^2\bar{\omega}_1^2 + 2|\omega_1|^2|\omega_2|^2)$$

hence

$$\Delta^2 = -\frac{1}{4}(\omega_1\bar{\omega}_2 - \omega_2\bar{\omega}_1)^2 = -D$$

We also note that given the quadratic form $An^2+2Bmn+Cm^2$ its discriminant being B^2-AC translates into $< \omega_1 \cdot \omega_2 >^2 - |\omega_1|^2|\omega_2|^2 = -D$

[12]To be pedantic, $\tau = -\overline{\tau_{CM}}$ if we want to insist that τ belongs to the upper half-plane. But in the ensuing discussion we can afford to be a bit sloppy.

[13]An example of such z is given by $z = \frac{\omega_2}{\omega_1}$ as

$$\begin{aligned} \tfrac{\omega_2}{\omega_1}\omega_1 &= \omega_2 \\ \tfrac{\omega_2}{\omega_1}\omega_2 &= c\omega_1 + d\omega_2 \end{aligned}$$

where

$$(\frac{\omega_2}{\omega_1})^2 = d\frac{\omega_2}{\omega_1} + c$$

is assumed to satisfy an integral quadratic equation. This will of course not be true in general, but whenever we note that $c = -\mathrm{Nm}(\tau), d = \mathrm{Tr}(\tau)$

[14]What we need to do is given D find t such that the discriminant of the equation x^2+tx+a is a square multiple of D i.e. $t^2 - 4A = k^2D$. If $D = -1$ it means that A needs to be the sum of two squares which is not always possible. Note that the norms of elements in $\mathbb{Q}[i]$ are obviously sums of two squares. However we have that $D = \frac{B^2-4AC}{A^2}$ if $B^2 - 4AC = -1$ we see that $A|B^2 + 1$ and hence A is a sum of two squares.

[15]To each quadratic field we may associate the ring of integral elements, namely those which satisfy a quadratic equation with integral coefficients and monic. To each quadratic field we get a discriminant by computing the discriminant of different elements, which will all differ by a square. This ring will have the property that all its elements leave the lattice Λ invariant under multiplication. The ring itself will be a rank two lattice spanned by $1, \tau$ whose trace is not necessarily even, hence it does not necessarily correspond to a quadratic form with even middle term, the kind favored by Gauss. More specifically. The full ring of integral elements will be elements of the type $\alpha\tau + \beta$ such that their traces and norms are integral. It is easy to compute

$$\mathrm{Tr}(\alpha\tau + \beta) = \alpha\mathrm{Tr}(\tau) + 2\beta \quad \mathrm{Nm}(\alpha\tau + \beta) = \alpha^2\mathrm{Nm}(\tau) + \alpha\beta\mathrm{Tr}(\tau) + \beta^2$$

From which we conclude that if $D \neq 3(4)$ then α, β need to be integers, so the ring coincides with the original. However if $D = 3(4)$ we can also have $\alpha = \beta = \frac{1}{2}(\mathbb{Z})$ and hence the ring is generated by $1, \frac{1}{2}(1+\tau)$ this will be an element with odd trace (and thus not contained in Gauss study). In general the endomorphism ring will just be a subring of the integral ring, and will be referred to as an order.

[16]For an ideal generated by $\alpha \in R$ we can take $\omega_1 = \alpha, \omega_2 = \alpha\tau$ and thus the associated quadratic form will not be primitive, but we can factor out $|\alpha|^2$.

[17]To be more precise, elements $\pm M$ give the same action, as Moebius tranformations, and we are really looking at $\mathrm{PSL}(2,\mathbb{Z})$ known as the modular group Γ

[18]The ring is not a unique factorization domain, we have $(1+\sqrt{5}i)(1-\sqrt{5}i) = 6 = 2 \cdot 3$ where all the elements are irreducible. It is this fact that serves as an inspiration for the construction of the ideal generated by $2, 1+\sqrt{5}i$.

[19]If the unit is 1 A.U. the distances from the sun for the planet p is given by $0.4 + 0.3 \cdot 2^m$ where $m = \infty$ corresponds to Mercury, $m = 0$ Venus $m = 1$ the Earth, $m = 2$ Mars, $m = 4$ Jupiter, $m = 5$ Saturn and $m = 6$ Uranus, thus Ceres (and a host of others) would correspond to $m = 3$, the law breaks down for Neptune.

[20]He anticipated Legendre who resented that Gauss claimed priority, could nothing be left for him?

[21]This was later introduced in 1965 unawares that Gauss had used it a century and a half earlier.

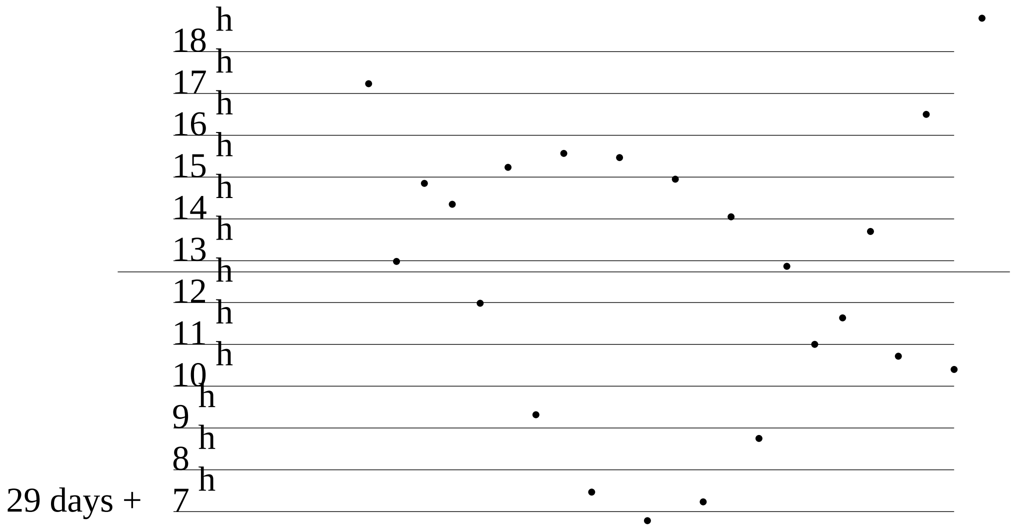
[22]'Rather death than such a life' actually discovered by Felix Klein

[23]In fact the ancients did it step by step, and as to be seen below this did not require much mathematical maturity, the Babylonians could have done it had they had occasion to do so. Apparently Gauss became interested in it, as his mother did not know the date of his birth, only the year and that it had been on Ascension Day.

[24]For some reason the theoretical possibility of April 26 is excluded on an *ad hoc* basis.

[25]In the culture of India, having lived through one thousand Full Moons is a significant event, which we see occurs around the time you turn 75.

[26]The standard time used nowadays is UTC which has replaced GMT. There was a Full Moon on November on November 14 at 13:52 and one on December 14 at 00:06 this corresponds to 29 days 10 hrs 14 min, the next one will be on January 12 at 11:34 UTC giving 29 days 11 hrs 28 min, subsequent periods will be 29 days 11 hrs 59 min, 29 days 14 hrs 21 min 29 days 15 hrs 14 min, thus we see that the Synodic period varies rater substantially. The average for those makes up 29 days 12 hrs 39 min to be compared with the long term average of 29 days 12 h 44 min (there is an advantage of having a virtual moon with some regularity compared to a real one). The first Full Moon after the vernal equinox on 2017 will be on Tuesday April 11 at 6:08 UTC making Sunday April 16, the next Easter Sunday. As to a Full Moon in the period in 2018 we have to subtract $365.24 - 354.36 = 10.88$ days from April 11 + 0.25 ending up on March 31 + 0.37 (give or take a few hours because of the irregularity) this will be a Saturday so Easter Sunday on 2018 will be on April 1. Below is shown the sybodic periods counted from full moon to full moon as well as new moon to new moon for about a years worth of phase cycles



[27]One such result was that in a non-euclidean geometry there would be an absolute unit of length, thus it would not be scale invariant, furthermore there would be a uniform bound on the areas of triangles, no matter how large.

[28]To another correspondent he wrote that he found Bolyai to be a genius of the first order

[29]the first one such measured was by Bessel in 1838 of 61 Cygni. Actually a binary star of high magnitude (5.2+6) just discernible to the naked eye. Rather unremarkable until Piazzi (the discoverer of Ceres) noted in 1804 that it had a large proper motion, i.e. that its position on the celestial square changed rapidly, indicating that it might be a close star. Bessel's estimate was 10.3 light years, rather close to the modern estimate of 11.4 . Of all stars visible to the naked eye it has the largest motion amounting to $4''$/year in Right Ascension and $3''$/year in Declination. At a declination of 40^o this amounts to about $4.5''$/year to be compared with that of Barnard's arrow star of 10"/year.

[30]Those are analogous to the case of spherical geometry. They were formally produced already in the 18th century by Lambert.

[31]Those were known already to Lambert in 1768 and were used by him and Legendre to prove the irrationality of π.

[32]We can see this in two different ways. Heuristically truncating an expansion starting with the term x^n gives for small values ϵ an error of at least the order of ϵ^n. If two power series start to differ at that term they will differ by that amount, but if the order of error is of a higher power, we get a contradiction. Another mathematically more satisfying way is to observe that the formula above shows that if x^n divides the power series $a_n(x) - b_n(x)$ then x^{2n} divides $a_{n+1}(x) - b_{n+1}(x)$.

[33]Using factorials or (integral) binomial coefficients we can of course put this into closed form. As Newton pointed out the power series expansion of $(1+a)^{\frac{1}{2}}$ is given as $\sum_k \binom{\frac{1}{2}}{k} a^k$ where the coefficients $(k = n+1)$ can be expanded out as

$$\frac{(\frac{1}{2})(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)\dots(\frac{1}{2}-n)}{(n+1)!} = (-1)^n \frac{1}{2^{n+1}} \frac{1\cdot 3\cdot 5\cdot \dots \cdot (2n-1)}{1\cdot 2\cdot 3\dots(n+1)}$$

where in the right hand side the even factors are missing. Filling them in $(n-1)$ of them we get

$$\frac{1}{2^{2n}} \frac{(2n-1)!}{(n+1)!(n-1)!} = \frac{1}{2^{2n}} \frac{1}{n-1} \binom{2n-1}{n+1}$$

. We would like to observe that up to the initial negative power of two we have an integer. To show that it is more convenient to use the left hand side. Note that the order of the factor p in $(n+1)!$ is given by

$$\lfloor \frac{n+1}{p} \rfloor + \lfloor \frac{n+1}{p^2} \rfloor + \lfloor \frac{n+1}{p^3} \rfloor \dots$$

while for the numerator we are only interested in the odd multiples of p and its powers (which we can assume to be odd) let us denote their numbers by $[n/p]$ etc. If we can show that $[n/q] \geq \lfloor \frac{n+1}{q} \rfloor$ for each odd number q we will be done. Now set $n+1 = kq + r$ with $0 \leq r < q$ and $k = \lfloor \frac{n+1}{q} \rfloor$. From this we find $2n - 1 = 2kq + 2r - 3$ if $2r - 3 < 0$ we will lose an even multiple of q but this does not matter as we will still have k odd multiples of q left, namely $q, 3q, 5q, \dots (2k-1)q$. We will get an excess when $2r - 3 \geq q$ (note that this always happens when $q > (n+1)$) which means that p divides the numerator $(q = p^n)$. Recall the list of the first few coefficients (note that n correspond to $n+1$ above)

n	1	2	3	4	5	6	7	8
num	1	1	1	5	7	21	33	429
exp of 2	1	3	4	7	9	10	11	15

We will get a factor of 3 in the denominator when $n = 6, 7, 8(9)$ (or if that is failing when $n = 18, 19, \dots (27)$ etc) a factor of 5 if $n = 4(5)$, a factor of 7 if $n = 5, 6(7)$ a factor of 11 if $n = 7, 8, \dots 10(11)$ and a factor of 13 if $n = 8, 9 \dots 12(13)$ all of which are ratified by the examples above. As an example we may compute the coefficient for $n = 30$. Looking at the primes $5, 7 \dots 53$ and the prime powers $9, 27, 25, 49$ up to $57 = 2 \cdot 30 - 3$ we conclude that the only factors will be $11, 17, 19$ as well as the automatic factors $31, 37, 41, 43, 47, 53$ and for 49 we will get the factor 7. Taking the product we see that the numerator will be (up to sign) 125280277081421. As to the denominator, we first observe that there will always be a power of n as to the additional we need to look at the power of two dividing $n!$ which is $\lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{4} \rfloor + \lfloor \frac{n}{8} \rfloor + \cdots \leq n-1$ with equality iff n is a power of two. In general we get $n-k$ where k is the number of ones in the binary expansion of 2. In the case of $n = 30$ this is given by $k = 4$ hence the denominator is $2^{30+30-4} = 2^{56}$. In conclusion we can claim that power series whose coefficients belong to dyadic numbers, i.e. denominators only divisible by two, are not only closed under addition and multiplication but also under taking square roots.

[34]A case in point is the computation of square roots by successive approximations, something known to the ancients, and a special case of Newton's general method. Let us say that we have an approximation x_n to $\sqrt{A}$ and we want to find a better one x_{n+1} by adding to x_n a slight disturbance ϵ. We get $(x_n+\epsilon)^2 = x_n^2 + 2x_n\epsilon + \epsilon^2 = A$ ignoring ϵ^2 we solve for ϵ getting $\epsilon = \frac{A-x_n^2}{2x_n}$ and adding to x_n that

$$x_{n+1} = \frac{A+x_n^2}{2x_n}$$

. This gives a rapid convergence as

$$A - x_{n+1}^2 = -\frac{(A-x_n^2)^2}{4x_n^2} \sim -\frac{1}{4A}(A-x_n^2)^2$$

(We may not that if $(A-x_n^2) > 4A$ we may not get any improvement but we eventually are rewarded). We may put $A = 2$ and $x_0 = 1$ and getting successively $\frac{3}{2}, \frac{17}{12}, \frac{577}{408} \ldots$ fractions $\frac{p_n}{q_n}$ satisfying the Pell's equation $p_n^2 - 2q_n^2 = 1$ readily verified. Likewise for $A = 3, x_0 = 1$ we get the series $2, \frac{7}{4}\frac{97}{56} \ldots$ where the fractions now satisfies $p_n^2 - 3q_n^2 = 1$. We may write down a table of successive approximations

A	x_1	x_2	x_3	x_4	x_5	x_6
2	1.500000	1.416667	1.414216	1.414214	1.414214	1.414214
3	2.000000	1.750000	1.732143	1.732051	1.732051	1.732051
4	2.500000	2.050000	2.000610	2.000000	2.000000	2.000000
5	3.000000	2.333333	2.238095	2.236069	2.236068	2.236068
6	3.500000	2.607143	2.454256	2.449494	2.449490	2.449490
7	4.000000	2.875000	2.654891	2.645767	2.645751	2.645751
8	4.500000	3.138889	2.843781	2.828469	2.828427	2.828427
9	5.000000	3.400000	3.023529	3.000092	3.000000	3.000000
10	5.500000	3.659091	3.196005	3.162456	3.162278	3.162278

which shows the rapid convergence.

We can also apply the same procedure to $A = 1+x$ starting with $x_0 = 1$ so that $A - x_0^2 = x$, thus x_1 should be correct up to the quadratic terms and we get

$$x_1 = \frac{(1+x)+1}{2} = 1 + \frac{x}{2}$$

as to x_2 it should be correct to terms up to x^4 and we obtain

$$x_2 = \frac{(1+x)+(1+\frac{x}{2})^2}{2(1+\frac{x}{2})} = \frac{1+x+\frac{x^2}{8}}{1+\frac{x}{2}}$$

where we expand the right hand side as

$$(1+x+\frac{x^2}{8})(1-\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{16}\ldots)$$

multiplying out we obtain

$$1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{x^4}{32}+\ldots$$

where terms up to x^4 are indeed correct but the x^4 term is not. The next step will be a true mess but the patient reader may convince herself that terms will be correct up to x^8 a truly painful way of obtaining the binomial theorem for $(1+x)^{\frac{1}{2}}$

[35]Jacobi supplies some of the details. An excellent exposition can be found in *The Arithmetic-Geometric Mean of Gauss* by David Cox in *L'enseignement Matheématique* IIe Série , Tome XXX -Fascicule 3-4, 1984 from which much of the discussion is lifted.

[36]cgs standing for centimeter, gram and second. In the more modern SI system one speaks abous Tesla (T) with the conversion $1T = 10^4 G$.

[37]If the quadric is given by $x^2 + k^2y^2$ we see that for any value of the parameter λ the curves $(x(t), y(t)) = \lambda(e^t, e^{k^2t})$ are integral curves, as the derivatives $\lambda(e^t, k^2e^{k^2t})$ have the same directions as the gradient at any point. The curves are in general transcendental, i.e. not satisfying any algebraic equation, but nevertheless they can be given a simple equation of type $y = \mu x^{k^2}$ with $\mu = \lambda^{k^2-1}$.

[38]recall that

$$(\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}dx)(\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}dy) = \iint_{\mathbb{R}^2} e^{-\frac{1}{2}(x^2+y^2)}dxdy = \int_0^{2\pi}\int_0^{\infty} e^{-\frac{1}{2}R^2}Rd\theta dr = 2\pi[-e^{-\frac{1}{2}R^2}]_0^{\infty} = 2\pi$$

[39]We note that setting $k = x\sqrt{2n}/2$ we have

$$\psi_{2n}(x) = \frac{\sqrt{2n}}{2}\frac{1}{2^{2n}}\binom{2n}{n+k} \sim \frac{\sqrt{4\pi n}\sqrt{2n}\cdot n^{2n}}{2\cdot\sqrt{4\pi^2(n-k)(n+k)}\cdot(n+k)^{n+k}(n-k)^{n-k}}$$

Now taking the log of $(\frac{n+k}{n})^{n+k}(\frac{n-k}{n})^{n-k}$ we get

$$(n+k)(\frac{k}{n} - \frac{1}{2}\frac{k^2}{n^2} + \dots) - (n-k)(\frac{k}{n} + \frac{1}{2}\frac{k^2}{n^2} + \dots) = \frac{2k^2}{n} - 2\frac{1}{2}\frac{k^2}{n} = \frac{k^2}{n}$$

as $k^2 = x^2\frac{2n}{4}$ we get $\frac{k^2}{n} = \frac{1}{2}x^2$ and hence the limit $\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$

[40]If $x = 1$ in the standard distribution, we talk about one standard deviation. This corresponds to the area $\frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-\frac{1}{2}x^2}dx = 0.682689\dots$. More generally this will be true for any Gaussian distribution, when 1 is replaced by σ one standard deviation. Thus the smaller standard deviation, the more concentrated around the average is the distribution.