COUNTING OCCURRENCES OF A PATTERN OF TYPE $(1,2)$ OR $(2,1)$ IN PERMUTATIONS

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ABSTRACT. Babson and Steingrímsson introduced generalized permutation patterns that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation. Claesson presented a complete solution for the number of permutations avoiding any single pattern of type $(1,2)$ or $(2,1)$. For eight of these twelve patterns the answer is given by the Bell numbers. For the remaining four the answer is given by the Catalan numbers.

With respect to being equidistributed there are three different classes of patterns of type $(1,2)$ or $(2,1)$. We present a recursion for the number of permutations containing exactly one occurrence of a pattern of the first or the second of the aforementioned classes, and we also find an ordinary generating function for these numbers. We prove these results both combinatorially and analytically. Finally, we give the distribution of any pattern of the third class in the form of a continued fraction, and we also give explicit formulas for the number of permutations containing exactly r occurrences of a pattern of the third class when $r \in \{1,2,3\}$.

1. INTRODUCTION AND PRELIMINARIES

Let $[n] = \{1,2,\ldots,n\}$ and denote by $\mathcal{S}_n$ the set of permutations of $[n]$. We shall view permutations in $\mathcal{S}_n$ as words with n distinct letters in $[n]$.

Classically, a pattern is a permutation $\sigma \in \mathcal{S}_k$, and an occurrence of σ in a permutation $\pi = a_1a_2\cdots a_n \in \mathcal{S}_n$ is a subword of π that is order equivalent to σ. For example, an occurrence of 132 is a subword $a_ia_ja_k$ $(1 \leq i < j < k \leq n)$ of π such that $a_i < a_k < a_j$. We denote by $s_\sigma^r(n)$ the number of permutations in $\mathcal{S}_n$ that contain exactly r occurrences of the pattern σ.

In the last decade much attention has been paid to the problem of finding the numbers $s_\sigma^r(n)$ for a fixed $r \geq 0$ and a given pattern σ (see [1, 2, 4, 6, 7, 8, 11, 13, 14, 16, 17, 18, 19, 20, 21]). Most of the authors consider only the case $r = 0$, thus studying permutations *avoiding* a given pattern. Only a few papers consider the case $r > 0$, usually restricting themselves to patterns of length 3. Using two simple involutions (*reverse* and *complement*) on $\mathcal{S}_n$ it is immediate that with respect to being equidistributed, the six patterns of length three fall into the two classes $\{123, 321\}$ and $\{132, 213, 231, 312\}$. Noonan [15] proved that $s_{123}^1(n) = \frac{3}{n}\binom{2n}{n-3}$. A general approach to the problem was suggested by Noonan and Zeilberger [16]; they gave another proof of Noonan's result, and conjectured that

$$s_{123}^2(n) = \frac{59n^2+117n+100}{2n(2n-1)(n+5)}\binom{2n}{n-4}$$

and $s_{132}^1(n) = \binom{2n-3}{n-3}$. The latter conjecture was proved by Bóna in [7]. A conjecture of Noonan and Zeilberger states that $s_\sigma^r(n)$ is P-recursive in n for any r and σ. It was proved by Bóna [5] for $\sigma = 132$.

Mansour and Vainshtein [14] suggested a new approach to this problem in the case $\sigma = 132$, which allows one to get an explicit expression for $s_{132}^r(n)$ for any given r.

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More precisely, they presented an algorithm that computes the generating function $\sum_{n\geq 0} s_{132}^r(n)x^n$ for any $r \geq 0$. To get the result for a given r, the algorithm performs certain routine checks for each element of the symmetric group S_{2r}. The algorithm has been implemented in C, and yields explicit results for $1 \leq r \leq 6$.

In [3] Babson and Steingrímsson introduced generalized permutation patterns that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation. The motivation for Babson and Steingrímsson in introducing these patterns was the study of Mahonian permutation statistics. Two examples of (generalized) patterns are 1-32 and 13-2. An occurrence of 1-32 in a permutation $\pi = a_1a_2\cdots a_n$ is a subword $a_ia_ja_{j+1}$ of π such that $a_i < a_{j+1} < a_j$. Similarly, an occurrence of 13-2 is a subword $a_ia_{i+1}a_j$ of π such that $a_i < a_j < a_{i+1}$. More generally, if $xyz \in \mathcal{S}_3$ and $\pi = a_1a_2\cdots a_n \in \mathcal{S}_n$, then we define

$$(x\text{-}yz)\,\pi \;=\; |\{a_ia_ja_{j+1} : \mathrm{proj}(a_ia_ja_{j+1}) = xyz, 1 \leq i < j < n\}|,$$

where $\mathrm{proj}(x_1x_2x_3)(i) = |\{j \in \{1,2,3\} : x_j \leq x_i\}|$ for $i \in \{1,2,3\}$ and $x_1, x_2, x_3 \in [n]$. For instance, $\mathrm{proj}(127) = \mathrm{proj}(138) = \mathrm{proj}(238) = 123$, and

$$(1\text{-}23)\,491273865 = |\{127, 138, 238\}| = 3.$$

Similarly, we also define $(xy\text{-}z)\,\pi = (z\text{-}yx)\,\pi^r$, where π^r denotes the reverse of π, that is, π read backwards.

For any word (finite sequence of letters), w, we denote by $|w|$ the length of w, that is, the number of letters in w. A pattern $\sigma = \sigma_1\text{-}\sigma_2\text{-}\cdots\text{-}\sigma_k$ containing exactly $k-1$ dashes is said to be of type $(|\sigma_1|, |\sigma_2|, \ldots, |\sigma_k|)$. For example, the pattern 142-5-367 is of type $(3,1,3)$, and any classical pattern of length k is of type $(\underbrace{1,1,\ldots,1}_{k})$.

In [11] Elizalde and Noy presented the following theorem regarding the distribution of the number of occurrences of any pattern of type (3).

Theorem 1 (Elizalde and Noy [11]). *Let $h(x) = \sqrt{(x-1)(x+3)}$. Then*

$$\sum_{\pi\in\mathcal{S}} x^{(123)\pi}\frac{t^{|\pi|}}{|\pi|!} = \frac{2h(x)e^{\frac{1}{2}(h(x)-x+1)t}}{h(x)+x+1+(h(x)-x-1)e^{h(x)t}},$$

$$\sum_{\pi\in\mathcal{S}} x^{(213)\pi}\frac{t^{|\pi|}}{|\pi|!} = \frac{1}{1-\int_0^t e^{(x-1)z^2/2}dz}.$$

The easy proof of the following proposition can be found in [9].

Proposition 2 (Claesson [9]). *With respect to being equidistributed, the twelve patterns of type $(1,2)$ or $(2,1)$ fall into the three classes*

$$\begin{aligned}&\{\ 1\text{-}23,\ 3\text{-}21,\ 12\text{-}3,\ 32\text{-}1\ \},\\ &\{\ 1\text{-}32,\ 3\text{-}12,\ 21\text{-}3,\ 23\text{-}1\ \},\\ &\{\ 2\text{-}13,\ 2\text{-}31,\ 13\text{-}2,\ 31\text{-}2\ \}.\end{aligned}$$

In the subsequent discussion we refer to the classes of the proposition above (in the order that they appear) as Class 1, 2 and 3 respectively.

Claesson [9] also gave a solution for the number of permutations avoiding any pattern of the type $(1,2)$ or $(2,1)$ as follows.

Proposition 3 (Claesson [9]). *Let $n \in \mathbb{N}$. We have*

$$|\mathcal{S}_n(\sigma)| = \begin{cases} B_n & \text{if } \sigma \in \{1\text{-}23, 3\text{-}21, 12\text{-}3, 32\text{-}1, 1\text{-}32, 3\text{-}12, 21\text{-}3, 23\text{-}1\},\\ C_n & \text{if } \sigma \in \{2\text{-}13, 2\text{-}31, 13\text{-}2, 31\text{-}2\},\end{cases}$$

where B_n and C_n are the nth Bell and Catalan numbers, respectively.

In particular, since B_n is not P-recursive in n, this result implies that for generalized patterns the conjecture that $s_\sigma^r(n)$ is P-recursive in n is false for $r = 0$ and, for example, $\sigma = 1\text{-}23$.

This paper is organized as follows. In Section 2 we find a recursion for the number of permutations containing exactly one occurrence of a pattern of Class 1, and we also find an ordinary generating function for these numbers. We prove these results both combinatorially and analytically. Similar results are also obtained for patterns of Class 2. In Section 3 we give the distribution of any pattern of Class 3 in the form of a continued fraction, and we also give explicit formulas for the number of permutations containing exactly r occurrences of a pattern of Class 3 when $r \in \{1, 2, 3\}$.

2. Counting occurrences of a pattern of Class 1 or 2

Theorem 4. *Let $u_1(n)$ be the number of permutations of length n containing exactly one occurrence of the pattern 1-23 and let B_n be the nth Bell number. The numbers $u_1(n)$ satisfy the recurrence*

$$u_1(n+2) = 2u_1(n+1) + \sum_{k=0}^{n-1} \binom{n}{k} \big[u_1(k+1) + B_{k+1}\big],$$

whenever $n \geq -1$, with the initial condition $u_1(0) = 0$.

Proof. Each permutation $\pi \in \mathcal{S}_{n+2}^1(1\text{-}23)$ contains a unique subword abc such that $a < b < c$ and bc is a segment of π. Let x be the last letter of π and define the sets $\mathcal{T}$, $\mathcal{T}'$, and $\mathcal{T}''$ by

$$\pi \in \begin{cases} \mathcal{T} & \text{if } x = 2, \\ \mathcal{T}' & \text{if } x \neq 2 \text{ and } a = 1, \\ \mathcal{T}'' & \text{if } x \neq 2 \text{ and } a \neq 1. \end{cases}$$

Then $\mathcal{S}_{n+2}^1(1\text{-}23)$ is the disjoint union of $\mathcal{T}$, $\mathcal{T}'$, and $\mathcal{T}''$, so

$$u_1(n+2) = |\mathcal{T}| + |\mathcal{T}'| + |\mathcal{T}''|.$$

Since removing/adding a trailing 2 from/to a permutation does not affect the number of hits of 1-23, we immediately get

$$|\mathcal{T}| = u_1(n+1).$$

For the cardinality of $\mathcal{T}'$ we observe that if $x \neq 2$ and $a = 1$ then $b = 2$: If the letter 2 precedes the letter 1 then every hit of 1-23 with $a = 1$ would cause an additional hit of 1-23 with $a = 2$ contradicting the uniqueness of the hit of 1-23; if 1 precedes 2 then $a = 1$ and $b = 2$. Thus we can factor any permutation $\pi \in \mathcal{T}'$ uniquely in the form $\pi = \sigma 2 \tau$, where σ is (1-23)-avoiding, the letter 1 is included in σ, and τ is nonempty and (12)-avoiding. Owing to Proposition 3 we have showed

$$|\mathcal{T}'| = \sum_{k=0}^{n-1} \binom{n}{k} B_{k+1}.$$

Suppose $\pi \in \mathcal{T}''$. Since $x \neq 2$ and $a \neq 1$ we can factor π uniquely in the form $\pi = \sigma 1 \tau$, where σ contains exactly one occurrence of 1-23, the letter 2 is included in σ, and τ is nonempty and (12)-avoiding. Consequently,

$$|\mathcal{T}''| = \sum_{k=0}^{n} \binom{n}{k} u_1(k+1),$$

which completes the proof. □

Example 5. Let us consider all permutations of length 5 that contain exactly one occurrence of 1-23, and give a small illustration of the proof of Theorem 12. If $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}''$ are defined as above then

$$\mathcal{T} = \underline{135}4|2 \quad \underline{1}4\underline{35}|2 \quad \underline{145}3|2 \quad \underline{1}5\underline{34}|2 \quad 4\underline{135}|2 \quad 5\underline{134}|2 \quad \underline{345}1|2$$

$$\mathcal{T}' = \begin{array}{llllll} \underline{1}|\underline{25}43 & \underline{1}3|\underline{25}4 & \underline{1}4|\underline{25}3 & \underline{1}43|\underline{25} & \underline{1}5|\underline{24}3 & \underline{1}53|\underline{24} \\ \underline{1}54|\underline{23} & 3\underline{1}|\underline{25}4 & 3\underline{1}4|\underline{25} & 3\underline{1}5|\underline{24} & 34\underline{1}|\underline{25} & 35\underline{1}|\underline{24} \\ 4\underline{1}|\underline{25}3 & 4\underline{1}3|\underline{25} & 4\underline{1}5|\underline{23} & 43\underline{1}|\underline{25} & 45\underline{1}|\underline{23} & 5\underline{1}|\underline{24}3 \\ 5\underline{1}3|\underline{24} & 5\underline{1}4|\underline{23} & 53\underline{1}|\underline{24} & 54\underline{1}|\underline{23} & & \end{array}$$

$$\mathcal{T}'' = \begin{array}{lllll} \underline{234}|15 & \underline{235}|14 & \underline{235}4|1 & \underline{2}4\underline{35}|1 & \underline{245}|13 \\ \underline{245}3|1 & \underline{2}5\underline{34}|1 & \underline{345}2|1 & 4\underline{235}|1 & 5\underline{234}|1 \end{array}$$

where the underlined subword is the unique hit of 1-23, and the bar indicates how the permutation is factored in the proof of Theorem 12.

Theorem 6. *Let $v_1(n)$ be the number of permutations of length n containing exactly one occurrence of the pattern* 1-32 *and let B_n be the nth Bell number. The numbers $v_1(n)$ satisfy the recurrence*

$$v_1(n+1) = v_1(n) + \sum_{k=1}^{n-1} \left[\binom{n}{k} v_1(k) + \binom{n-1}{k-1} B_k \right],$$

whenever $n \geq 0$, with the initial condition $v_1(0) = 0$.

Proof. Each permutation $\pi \in \mathcal{S}^1_{n+2}(1\text{-}32)$ contains a unique subword acb such that $a < b < c$ and cb is a segment of π. Define the sets $\mathcal{T}$ and $\mathcal{T}'$ by

$$\pi \in \begin{cases} \mathcal{T} & \text{if } a = 1, \\ \mathcal{T}' & \text{if } a \neq 1. \end{cases}$$

Then $\mathcal{S}^1_{n+2}(1\text{-}32)$ is the disjoint union of $\mathcal{T}$ and $\mathcal{T}'$, so

$$v_1(n+2) = |\mathcal{T}| + |\mathcal{T}'|.$$

For the cardinality of $\mathcal{T}$ we observe that if $a = 1$ then $b = 2$: If the letter 2 precedes the letter 1 or 12 is a segment of π then every hit of 1-23 with $a = 1$ would cause an additional hit of 1-32 with $a = 2$ contradicting the uniqueness of the hit of 1-23; if 1 precedes 2 then $a = 1$ and $b = 2$. Thus we can factor π uniquely in the form $\pi = \sigma x 2 \tau$, where σx is (1-32)-avoiding, the letter 1 is included in σ, and τ is nonempty and (12)-avoiding. Let $\mathcal{R}_n$ be the set of (1-32)-avoiding permutations of $[n]$ that do not end with the letter 1. Since the letter 1 cannot be the last letter of a hit of 1-32, we have, by Proposition 3, that $|\mathcal{S}^0_n(1\text{-}32) \setminus \mathcal{R}_n| = B_{n-1}$. Consequently, $|\mathcal{R}_n| = B_n - B_{n-1}$ and

$$\begin{aligned} |\mathcal{T}| &= \sum_{k=1}^{n} \binom{n-1}{k-1} |\mathcal{R}_k| \\ &= \sum_{k=1}^{n} \binom{n-1}{k-1} (B_k - B_{k-1}) \\ &= \sum_{k=1}^{n-1} \binom{n-1}{k-1} B_k. \end{aligned}$$

For the last identity we have used the familiar recurrence relation $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$.

Suppose $\pi \in \mathcal{T}'$. Since $a \neq 1$ we can factor π uniquely in the form $\pi = \sigma 1 \tau$, where σ contains exactly one occurrence of 1-32, and τ is nonempty and (12)-avoiding. Accordingly,

$$|\mathcal{T}''| = \sum_{k=0}^{n} \binom{n}{k} v_1(k),$$

which completes the proof. □

Let σ be a pattern of Class 1 or 2. Using combinatorial reasoning we have found a recursion for the number of permutations containing exactly one occurrence of the pattern σ (Theorem 4 and 6). More generally, given $r \geq 0$, we would like to find a recursion for the number of permutations containing exactly r occurrence of the pattern σ. Using a more general and analytic approach we will now demonstrate how this (at least in principle) can be achieved.

Let $S_\sigma^r(x)$ be the generating function $S_\sigma^r(x) = \sum_n s_\sigma^r(n)x^n$. To find functional relations for $S_\sigma^r(x)$ the following lemma will turn out to be useful.

Lemma 7. *If $\{a_n\}$ is a sequence of numbers and $A(x) = \sum_{n\geq 0} a_n x^n$ is its ordinary generating function, then, for any $d \geq 0$,*

$$\sum_{n\geq 0}\left[\sum_{j=0}^{n}\binom{n}{j}a_{j+d}\right]x^n = \frac{(1-x)^{d-1}}{x^d}\left[A\Big(\frac{x}{1-x}\Big) - \sum_{j=0}^{d-1} a_j \Big(\frac{x}{1-x}\Big)^j\right].$$

Proof. It is plain that

$$\sum_{n\geq 0}\Big[\sum_{j=0}^{n}\binom{n}{j}a_j\Big]x^n = \frac{1}{1-x}A\Big(\frac{x}{1-x}\Big).$$

See for example [12, p 192]. On the other hand,

$$\sum_{n\geq 0} a_{n+d}x^n = \frac{1}{x^d}\Big[A(x) - \sum_{j=0}^{d-1} a_j x^j\Big].$$

Combining these two identities we get the desired result. □

Define $\mathcal{S}_n^r(\sigma)$ to be the set of permutations $\pi \in S_n$ such that $(\sigma)\pi = r$. Let $s_\sigma^r(n) = |\mathcal{S}_n^r(\sigma)|$ for $r \geq 0$ and $s_\sigma^r(n) = 0$ for $r < 0$. Given $b_1, b_2, \ldots, b_k \in \mathbb{N}$, we also define

$$s_\sigma^r(n; b_1, b_2, \ldots, b_k) = \#\{a_1a_2\cdots a_n \in \mathcal{S}_n^r(\sigma) \mid a_1a_2\cdots a_k = b_1b_2\cdots b_k\}.$$

As a direct consequence of the above definitions, we have

$$s_\sigma^r(n) = \sum_{j=1}^{n} s_\sigma^r(n;j). \tag{1}$$

We start by considering patterns that belong to Class 1 and we use 12-3 as a representative of this class. Let us define

$$\begin{aligned} u_r(n; b_1, \ldots, b_k) &= s_{12\text{-}3}^r(n; b_1, \ldots, b_k), \\ u_r(n) &= s_{12\text{-}3}^r(n), \\ U_r(x) &= S_{12\text{-}3}^r(x). \end{aligned}$$

Lemma 8. *Let $n \geq 1$. We have $u_r(n; n-1) = u_r(n; n) = u_r(n-1)$ and*

$$u_r(n;i) = \sum_{j=1}^{i-1} u_r(n-1;j) + \sum_{j=0}^{n-i-1} u_{r-j}(n-1; n-1-j),$$

whenever $1 \leq i \leq n-2$.

Proof. If $a_1 a_2 \cdots a_n$ is any permutation of $[n]$ then

$$(12\text{-}3)a_1 a_2 \cdots a_n = (12\text{-}3)a_2 a_3 \cdots a_n + \begin{cases} n - a_2 & \text{if } a_1 < a_2, \\ 0 & \text{if } a_1 > a_2. \end{cases}$$

Hence,

$$\begin{aligned} u_r(n;i) &= \sum_{j=1}^{i-1} u_r(n;i,j) + \sum_{j=i+1}^{n} u_r(n;i,j) \\ &= \sum_{j=1}^{i-1} u_r(n-1;j) + \sum_{j=i+1}^{n} u_{r-n+j}(n-1;j-1) \\ &= \sum_{j=1}^{i-1} u_r(n-1;j) + \sum_{j=0}^{n-i-1} u_{r-j}(n-1;n-1-j). \end{aligned}$$

For $i = n-1$ or $i = n$ it is easy to see that $u_r(n;i) = u_r(n-1)$. ☐

Using Lemma 8 we quickly generate the numbers $u_r(n)$; the first few of these numbers are given in Table 1. Given $r \in \mathbb{N}$ we can also use Lemma 8 to find a

$n \backslash r$	0	1	2	3	4	5	6
0	1						
1	1						
2	2						
3	5	1					
4	15	7	1	1			
5	52	39	13	12	2	1	1
6	203	211	112	103	41	24	17
7	877	1168	843	811	492	337	238
8	4140	6728	6089	6273	4851	3798	2956
9	21147	40561	43887	48806	44291	38795	33343
10	115975	256297	321357	386041	394154	379611	355182

TABLE 1. The number of permutations of length n containing exactly r occurrences of the pattern 12-3.

functional relation determining $U_r(x)$. Here we present such functional relations for $r = 0, 1, 2$ and also explicit formulas for $r = 0, 1$.

Equation 1 tells us how to compute $u_r(n)$ if we are given the numbers $u_r(n;i)$. For the case $r = 0$ Lemma 9, below, tells us how to do the converse.

Lemma 9. *If* $1 \leq i \leq n-2$ *then*

$$u_0(n;i) = \sum_{j=0}^{i-1} \binom{i-1}{j} u_0(n-2-j).$$

Proof. For $n = 1$ the identity is trivially true. Assume the identity is true for $n = m$. We have

$$\begin{aligned} u_0(m+1;i) &= \sum_{j=1}^{i-1} u_0(m;j) + u_0(m-1) && \text{by Lemma 8} \\ &= \sum_{j=1}^{i-1}\sum_{k=0}^{j-1} \binom{j-1}{k} u_0(m-2-k) + u_0(m-1) && \text{by the induction hypothesis} \\ &= \sum_{j=1}^{i-1}\sum_{k=j-1}^{i-2} \binom{k}{j-1} u_0(m-1-j). \end{aligned}$$

Using the familiar equality $\binom{1}{k} + \binom{2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$ we then get

$$u_0(m+1;i) = \sum_{j=1}^{i-1} \binom{i-1}{j} u_0(m-1-j).$$

Thus the identity is true for $n = m+1$ and by the principle of induction the desired identity is true for all $n \geq 1$. $\square$

The following proposition is a direct consequence of Proposition 3. However, we give a different proof. The proof is intended to illustrate the general approach. It is advisable to read this proof before reading the proof of Theorem 4′ below.

Proposition 10. *The ordinary generating function for the number of* (12-3)*-avoiding permutations of length* n *is*

$$U_0(x) = \sum_{k\geq 0} \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}.$$

Proof. We have

$$\begin{aligned} u_0(n) &= \sum_{k=1}^{n} u_0(n;k) && \text{by Equation 1} \\ &= 2u_0(n-1) + \sum_{i=1}^{n-2}\sum_{j=0}^{i-1} \binom{i-1}{j} u_0(n-2-j) && \text{by Lemma 8 and 9} \\ &= u_0(n-1) + \sum_{i=0}^{n-2} \binom{n-2}{i} u_0(n-1-i) && \text{by } \sum_{i=k}^{n}\binom{i}{k} = \binom{n+1}{k+1} \\ &= u_0(n-1) + \sum_{i=0}^{n-2} \binom{n-2}{i} u_0(i+1). \end{aligned}$$

Therefore, by Lemma 7, we have

$$U_0(x) = xU_0(x) + 1 - x + xU_0\left(\frac{x}{1-x}\right),$$

which is equivalent to

$$U_0(x) = 1 + \frac{x}{1-x} U_0\left(\frac{x}{1-x}\right).$$

An infinite number of applications of this identity concludes the proof. $\square$

We now derive a formula for $U_1(x)$ that is somewhat similar to the one for $U_0(x)$. The following lemma is a first step in this direction.

Lemma 11. *If* $1 \le i \le n-2$ *then*

$$u_1(n;i) = \sum_{j=0}^{i-1} \binom{i-1}{j} u_1(n-2-j) + u_0(n;i).$$

Proof. For $n = 1$ the identity is trivially true. Assume the identity is true for $n = m$. Lemma 8 and the induction hypothesis imply

$$\begin{aligned} u_1(m+1;i) &= \sum_{j=1}^{i-1} u_1(m;j) + u_1(m-1) + u_0(m-1) \\ &= \sum_{j=0}^{i-1} \binom{j-1}{k} u_1(m-1-j) + \sum_{j=1}^{i-1} u_0(m;j) + u_0(m-1). \end{aligned}$$

In addition, Lemma 9 implies

$$\begin{aligned} u_0(m+1;i) &= \sum_{j=1}^{i-1}\sum_{k=0}^{j-1} \binom{j-1}{k} u_0(n-2-k) + u_0(n-1) \\ &= \sum_{j=0}^{i-1} \binom{i-1}{j} u_0(n-1-j) \\ &= \sum_{j=1}^{i-1} u_0(m;j) + u_0(m-1). \end{aligned}$$

Thus the identity is true for $n = m+1$ and by the principle of induction the desired identity is true for all $n \ge 1$. □

Next, we rediscover Theorem 4.

Theorem 4′. *Let* $u_1(n)$ *be the number of permutations of length* n *containing exactly one occurrence of the pattern* 12-3 *and let* B_n *be the* n*th Bell number. The numbers* $u_1(n)$ *satisfy the recurrence*

$$u_1(n+2) = 2u_1(n+1) + \sum_{k=0}^{n-1} \binom{n}{k} \big[u_1(k+1) + B_{k+1}\big],$$

whenever $n \ge -1$*, with the initial condition* $u_1(0) = 0$.

Proof. Similarly to the proof of Proposition 10, we use Equation 1, Lemma 8, 9, and 11 to get

$$\begin{aligned} u_1(n) &= 2u_1(n-1) + \sum_{i=1}^{n-2} \left[\sum_{j=0}^{i-1} \binom{i-1}{j} u_1(n-2-j) + u_0(n;i) \right] \\ &= 2u_1(n-1) + \sum_{i=1}^{n-2}\sum_{j=0}^{i-1} \binom{i-1}{j} \big(u_1(n-2-j) + u_0(n-2-j)\big) \\ &= u_1(n-1) - u_0(n-1) + \sum_{i=0}^{n-2} \binom{n-2}{i} \big(u_1(i+1) + u_0(i+1)\big) \\ &= 2u_1(n-1) + \sum_{i=0}^{n-3} \binom{n-2}{i} \big(u_1(i+1) + u_0(i+1)\big). \end{aligned}$$

□

Corollary 12. *The ordinary generating function, $U_1(x)$, for the number of permutations of length n containing exactly one occurrence of the pattern* 12-3 *satisfies the functional equation*

$$U_1(x) = \frac{x}{1-x}\left(U_1\left(\frac{x}{1-x}\right) + U_0\left(\frac{x}{1-x}\right) - U_0(x)\right).$$

Proof. The result follows from Theorem 4 together with Lemma 7. □

Corollary 13. *The ordinary generating function for the number of permutations of length n containing exactly one occurrence of the pattern* 12-3 *is*

$$U_1(x) = \sum_{n\geq 1} \frac{x}{1-nx} \sum_{k\geq 0} \frac{kx^{k+n}}{(1-x)(1-2x)\cdots(1-(k+n)x)}.$$

Proof. We simply apply Corollary 12 an infinite number of times and in each step we perform some rather tedious algebraic manipulations. □

Theorem 14. *The ordinary generating function, $U_2(x)$, for the number of permutations of length n containing exactly two occurrences of the pattern* 12-3 *satisfies the functional equation*

$$U_2(x) = \frac{x}{(1-x)^2(1-2x)}\left(\begin{aligned} &U_2\left(\frac{x}{1-x}\right) - (1-x)U_2(x) + \\ &U_1\left(\frac{x}{1-x}\right) - (1-x)^2 U_1(x) + \\ &U_0\left(\frac{x}{1-x}\right) - (1-x)^2 U_0(x) \end{aligned} \right).$$

Proof. The proof is similar to the proofs of Lemma 11, Theorem 4' and Corollary 12, and we only sketch it here.

Lemma 8 yields

$$\begin{aligned} u_2(n;n) &= u_2(n-1) \\ u_2(n;n-1) &= u_2(n-1) \\ u_2(n;n-2) &= u_2(n-1) - u_2(n-2) + u_1(n-2) \end{aligned}$$

and, by means of induction,

$$u_2(n;i) = u_1(n;i) + u_0(n;i) - u_0(n-1;i) + \sum_{j=0}^{i-1} \binom{i-1}{j} u_2(n-2-j),$$

whenever $1 \leq i \leq n-3$. Therefore, $u_2(0) = u_2(1) = u_2(2) = 0$ and

$$\begin{aligned} u_2(n) = {} & 3u_2(n-1) - u_2(n-2) + u_1(n-2) + \\ & \sum_{i=1}^{n-3} \binom{n-3}{i} (u_2(n-1-i) + u_1(n-1-i) + u_0(n-1-i) - u_0(n-2-i)). \end{aligned}$$

whenever $n \geq 3$. Thus, the result follows from Lemma 7. □

We now turn our attention to patterns that belong to Class 2 and we use 23-1 as a representative of this class. The results found below regarding the 23-1 pattern are very similar to the ones previously found for the 12-3 pattern, and so are the proofs; therefore we choose to omit most of the proofs. However, we give the necessary lemmas from which the reader may construct her/his own proofs.

Define

$$\begin{aligned} v_r(n;b_1,\ldots,b_k) &= s^r_{23\text{-}1}(n;b_1,\ldots,b_k), \\ v_r(n) &= s^r_{23\text{-}1}(n), \\ V_r(x) &= S^r_{23\text{-}1}(x). \end{aligned}$$

If $a_1a_2\cdots a_n$ is any permutation of $[n]$ then

$$(23\text{-}1)a_1a_2\cdots a_n = (23\text{-}1)a_2a_3\cdots a_n + \begin{cases} a_1-1 & \text{if } a_1<a_2, \\ 0 & \text{if } a_1>a_2. \end{cases}$$

Lemma 15. *Let $n\geq 1$. We have $v_r(n;1)=v_r(n;n)=v_r(n-1)$ and*

$$v_r(n;i)=\sum_{j=1}^{i-1} v_r(n-1;j)+\sum_{j=i}^{n-1} v_{r-i+1}(n-1;j),$$

whenever $2\leq i\leq n-1$.

Using Lemma 15 we quickly generate the numbers $v_r(n)$; the first few of these numbers are given in Table 2.

$n\backslash r$	0	1	2	3	4	5	6
0	1						
1	1						
2	2						
3	5	1					
4	15	6	3				
5	52	32	23	10	3		
6	203	171	152	98	62	22	11
7	877	944	984	791	624	392	240
8	4140	5444	6460	6082	5513	4302	3328
9	21147	32919	43626	46508	46880	41979	36774
10	115975	208816	304939	360376	396545	393476	377610

TABLE 2. The number of permutations of length n containing exactly r occurrences of the pattern 23-1.

Lemma 16. *If $2\leq i\leq n-1$ then*

$$v_0(n;i)=\sum_{j=0}^{i-2}\binom{i-2}{j} v_0(n-2-j).$$

Proposition 17. *The ordinary generating function for the number of $(23\text{-}1)$-avoiding permutations of length n is*

$$V_0(x)=\sum_{k\geq 0}\frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}.$$

Lemma 18. *If $2\leq i\leq n-1$ then*

$$v_1(n;i)=\sum_{j=0}^{i-2}\binom{i-2}{j} v_1(n-2-j)+v_0(n;i-1)-v_0(n-1,i-1).$$

Theorem 6′. *Let $v_1(n)$ be the number of permutations of length n containing exactly one occurrence of the pattern 23-1 and let B_n be the nth Bell number. The numbers $v_1(n)$ satisfy the recurrence*

$$v_1(n+1) = v_1(n) + \sum_{k=1}^{n-1}\left[\binom{n}{k}v_1(k) + \binom{n-1}{k-1}B_k\right],$$

whenever $n \geq 0$, with the initial condition $v_1(0) = 0$.

Corollary 19. *The ordinary generating function for the number of permutations of length n containing exactly one occurrence of the pattern 23-1 satisfies the functional equation*

$$V_1(x) = \frac{x}{1-x}V_1\Big(\frac{x}{1-x}\Big) + x\bigg(V_0\Big(\frac{x}{1-x}\Big) - V_0(x)\bigg).$$

Corollary 20. *The ordinary generating function for the number of permutations of length n containing exactly one occurrence of the pattern 23-1 is*

$$V_1(x) = \sum_{n\geq 1}\frac{x}{1-(n-1)x}\sum_{k\geq 0}\frac{kx^{k+n}}{(1-x)(1-2x)\cdots(1-(k+n)x)}.$$

Theorem 21. *The ordinary generating function, $V_2(x)$, for the number of permutations of length n containing exactly two occurrences of the pattern 23-1 satisfies the functional equation*

$$V_2(x) = \frac{x}{1-x}\bigg(V_2\Big(\frac{x}{1-x}\Big) + (1-2x)V_1\Big(\frac{x}{1-x}\Big) + (1-3x+x^2)V_0\Big(\frac{x}{1-x}\Big)\bigg) - x + x^2$$

Proof. By Lemma 5

$$\begin{aligned}
v_2(n;n) &= v_2(n-1)\\
v_2(n;1) &= v_2(n-1)\\
v_2(n;2) &= v_2(n-2) + v_1(n-1) - v_1(n-2)\\
v_2(n;3) &= v_2(n-2) + v_2(n-3) + v_1(n-2) - v_1(n-3) +\\
&\qquad + v_0(n-1) - v_0(n-2) - v_0(n-3)
\end{aligned}$$

and, by means of induction,

$$v_2(n;i) = \sum_{j=0}^{i-2}\binom{i-2}{j}v_2(n-2-j) + v_1(n;i-1) + v_1(n-1;i-1) - v_0(n-1;i-2)$$

for $n-1 \geq i \geq 4$. Thus $v_2(0) = v_2(1) = v_2(2) = 0$ and for all $n \geq 3$

$$\begin{aligned}
v_2(n) = v_2(n-1) &+ \sum_{j=0}^{n-2}\binom{n-2}{j}v_2(n-1-j) +\\
&+ \sum_{j=0}^{n-3}\binom{n-3}{j}\big(v_1(n-1-j) - v_1(n-2-j)\big) +\\
&+ \sum_{j=0}^{n-4}\binom{n-4}{j}\big(v_0(n-1-j) - v_0(n-2-j) - v_0(n-3-j)\big).
\end{aligned}$$

The result now follows from Lemma 7. ◻

3. Counting occurrences of a pattern of Class 3

We choose 2-13 as our representative for Class 3 and we define $w_r(n)$ as the number of permutations of length n containing exactly r occurrences of the pattern 2-13. We could apply the analytic approach from the previous section to the problem of determining $w_r(n)$. However, a result by Clarke, Steingrímsson and Zeng [10, Corollary 11] provides us with a better option.

Theorem 22. *The following Stieltjes continued fraction expansion holds*

$$\sum_{\pi\in\mathcal{S}} x^{1+(12)\pi} y^{(21)\pi} p^{(2\text{-}31)\pi} q^{(31\text{-}2)\pi} t^{|\pi|} = \cfrac{1}{1-\cfrac{x[\,1\,]_{p,q}t}{1-\cfrac{y[\,1\,]_{p,q}t}{1-\cfrac{x[\,2\,]_{p,q}t}{1-\cfrac{y[\,2\,]_{p,q}t}{\ddots}}}}}$$

where $[\,n\,]_{p,q} = q^{n-1} + pq^{n-2} + \cdots + p^{n-2}q + p^{n-1}$.

Proof. In [10, Corollary 11] Clarke, Steingrímsson and Zeng derived the following continued fraction expansion

$$\sum_{\pi\in\mathcal{S}} y^{\operatorname{des}\pi} p^{\operatorname{Res}\pi} q^{\operatorname{Ddif}\pi} t^{|\pi|} = \cfrac{1}{1-\cfrac{[\,1\,]_p t}{1-\cfrac{yq[\,1\,]_p t}{1-\cfrac{q[\,2\,]_p t}{1-\cfrac{yq^2[\,2\,]_p t}{\ddots}}}}}$$

where $[\,n\,]_p = 1 + p + \cdots + p^{n-1}$. We refer the reader to [10] for the definitions of Ddif and Res. However, given these definitions, it is easy to see that Res $=$ (2-31) and Ddif $= (21) + (2\text{-}31) + (31\text{-}2)$. Moreover, des $= (21)$ and $|\pi| = 1 + (12)\pi + (21)\pi$. Thus, substituting $y(xq)^{-1}$ for y, pq^{-1} for p, and xt for t, we get the desired result. □

The following corollary is an immediate consequence of Theorem 22.

Corollary 23. *The bivariate ordinary generating function for the distribution of occurrences of the pattern* 2-13 *admits the Stieltjes continued fraction expansion*

$$\sum_{\pi\in\mathcal{S}} p^{(2\text{-}13)\pi} t^{|\pi|} = \cfrac{1}{1-\cfrac{[\,1\,]_p t}{1-\cfrac{[\,1\,]_p t}{1-\cfrac{[\,2\,]_p t}{1-\cfrac{[\,2\,]_p t}{\ddots}}}}}$$

where $[\,n\,]_p = 1 + p + \cdots + p^{n-1}$

Using Corollary 23 we quickly generate the numbers $w_r(n)$; the first few of these numbers are given in Table 3.

Corollary 24. *The number of* (2-13)*-avoiding permutations of length* n *is*

$$w_0(n) = \frac{1}{n+1}\binom{2n}{n}.$$

$n\backslash r$	0	1	2	3	4	5	6
0	1						
1	1						
2	2						
3	5	1					
4	14	8	2				
5	42	45	25	7	1		
6	132	220	198	112	44	12	2
7	429	1001	1274	1092	700	352	140
8	1430	4368	7280	8400	7460	5392	3262
9	4862	18564	38556	56100	63648	59670	47802
10	16796	77520	193800	341088	470934	541044	535990

TABLE 3. The number of permutations of length n containing exactly r occurrences of the pattern 2-13.

Proof. This result is explicitly stated in Proposition 3, but it also follows from Corollary 23 by putting $p = 0$. □

Corollary 25. *The number of permutations of length* n *containing exactly one occurrence of the pattern* 2-13 *is*

$$w_1(n) = \binom{2n}{n-3}.$$

Proof. For $m > 0$ let

$$W(p,t;m) = \cfrac{1}{1 - \cfrac{[\,m\,]_p t}{1 - \cfrac{[\,m\,]_p t}{1 - \cfrac{[\,m+1\,]_p t}{1 - \cfrac{[\,m+1\,]_p t}{\ddots}}}}}$$

Note that

$$W(p,t;m) = \cfrac{1}{1 - \cfrac{[\,m\,]_p t}{1 - [\,m\,]_p t W(p,t;m+1).}}$$

Assume $m > 1$. Differentiating $W(p,t;m)$ with respect to p and evaluating the result at $p = 0$ we get

$$D_p W(p,t;m)\big|_{p=0} = tC(t)^3 + t^2C(t)^5 + t^2C(t)^4 D_p W(p,t;m+1)\big|_{p=0}$$

where $C(t) = W(0,t,1)$ is the generating function for the Catalan numbers. Applying this identity an infinite number of times we get

$$D_p W(p,t,m)\big|_{p=0} = tC(t)^3 + t^2C(t)^5 + t^3C(t)^7 + \cdots = \frac{tC(t)^3}{1 - tC(t)^2}.$$

On the other hand, $D_p W(p,t;1)\big|_{p=0} = t^2C(t)^4 D_p W(p,t;2)\big|_{p=0}$. Combining these two identities we get

$$D_p W(p,t;1)\big|_{p=0} = \frac{t^3C(t)^7}{1 - tC(t)^2}.$$

Since $\sum_{n\geq 0} w_1(n)t^n = D_pW(p,t;1)\big|_{p=0}$ the proof is completed on extracting coefficients in the last identity. $\square$

The proofs of the following two corollaries are similar to the proof of Corollary 25 and are omitted.

Corollary 26. *The number of permutations of length* n *containing exactly two occurrences of the pattern* 2-13 *is*

$$w_2(n) = \frac{n(n-3)}{2(n+4)}\binom{2n}{n-3}.$$

Corollary 27. *The number of permutations of length* n *containing exactly three occurrences of the pattern* 2-13 *is*

$$w_3(n) = \frac{1}{3}\binom{n+2}{2}\binom{2n}{n-5}.$$

As a concluding remark we note that there are many questions left to answer. What is, for example, the formula for $w_k(n)$ in general? What are the combinatorial explanations of $ns^1_{1\text{-}2\text{-}3}(n) = 3s^1_{2\text{-}13}(n)$ and

$$(n+3)(n+2)(n+1)s^1_{2\text{-}13}(n) = 2n(2n-1)(2n-2)s^1_{2\text{-}1\text{-}3}(n)?$$

In addition, Corollary 25 obviously is in need of a combinatorial proof.

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