A CHARACTERIZATION OF THE ODD BINOMIAL COEFFICIENTS

ANDERS CLAESSON

The aim of this short paper is to analyse Pascal's triangle modulo 2. Here follows an illustration of the first 16 rows of this triangle (■:=1, □:=0) :

0	■	0
1	■ ■	1
2	■ □ ■	2
3	■ ■ ■ ■	3
4	■ □ □ □ ■	4
5	■ ■ □ □ ■ ■	5
6	■ □ ■ □ ■ □ ■	6
7	■ ■ ■ ■ ■ ■ ■ ■	7
8	■ □ □ □ □ □ □ □ ■	8
9	■ ■ □ □ □ □ □ □ ■ ■	9
10	■ □ ■ □ □ □ □ □ ■ □ ■	10
11	■ ■ ■ ■ □ □ □ □ ■ ■ ■ ■	11
12	■ □ □ □ ■ □ □ □ ■ □ □ □ ■	12
13	■ ■ □ □ ■ ■ □ □ ■ ■ □ □ ■ ■	13
14	■ □ ■ □ ■ □ ■ □ ■ □ ■ □ ■ □ ■	14
15	■ ■ ■ ■ ■ ■ ■ ■ ■ ■ ■ ■ ■ ■ ■ ■	15

Take any $n \in \mathbb{N}$ and let $n = \sum_i b_i 2^i$, $b_i \in \{0,1\}$, be its binary expansion. Then we define $C_n = \{2^i : b_i = 1\}$, e.g. $13 = 1+4+8$ and hence $C_{13} = \{1,4,8\}$. Moreover we define $w(n)$ as the number of i's such that $b_i = 1$, e.g. $w(13) = w(1101_2) = 3$.

Proposition. *If $0 \le k \le n$ then $\binom{n}{k}$ is odd if and only if $C_k \subseteq C_n$.*

Proof. By an easy induction argument we see that $(1+x)^{2^m} \equiv 1+x^{2^m} \pmod{2}$ for every $m \in \mathbb{N}$. Now fix an $n \in \mathbb{N}$, by the binomial theorem $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ and hence $\binom{n}{k}$ is odd if and only if x^k is a term in $(1+x)^n$ when counting modulo two. Moreover

$$(1+x)^n = \prod_{c \in C_n} (1+x)^c \equiv \prod_{c \in C_n} (1+x^c) = \sum_{k : C_k \subseteq C_n} x^k \pmod{2}$$

which concludes the proof. □

Corollary. *There are exactly $2^{w(n)}$ odd entries in row n of Pascal's triangle.*

Proof. The polynomial $\displaystyle\sum_{k : C_k \subseteq C_n} x^k$ contains $|\mathcal{P}(C_n)| = 2^{|C_n|} = 2^{w(n)}$ terms. □

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