PARTIALLY ORDERED GENERALIZED PATTERNS AND k-ARY WORDS

Sergey Kitaev and Toufik Mansour [1]

Matematik, Chalmers tekniska högskola och Göteborgs universitet,
S-412 96 Göteborg, Sweden

kitaev@math.chalmers.se, toufik@math.chalmers.se

Abstract

Recently, Kitaev [Ki2] introduced partially ordered generalized patterns (POGPs) in the symmetric group, which further generalize the generalized permutation patterns introduced by Babson and Steingrímsson [BS]. A POGP p is a GP some of whose letters are incomparable. In this paper, we study the generating functions (g.f.) for the number of k-ary words avoiding some POGPs. We give analogues, extend and generalize several known results, as well as get some new results. In particular, we give the g.f. for the entire distribution of the maximum number of non-overlapping occurrences of a pattern p with no dashes (that allowed to have repetition of letters), provided we know the g.f. for the number of k-ary words that avoid p.

Keywords: words, generalized patterns, partially ordered generalized patterns, generating functions

2000 Mathematics Subject Classification: 05A05, 05A15

1. Introduction

Let $[k]^n$ denote the set of all the words of length n over the (totally ordered) alphabet $[k] = \{1, 2, \ldots, k\}$. We call these words by n-long k-ary words. A *generalized pattern* τ is a word in $[\ell]^m$ (possibly with dashes between some letters) that contains each letter from $[\ell]$ (possibly with repetitions). We say that the word $\sigma \in [k]^n$ *contains* a generalized pattern τ, if σ contains a subsequence isomorphic to τ in which the entries corresponding to consecutive entries of τ, which are not separated by a dash, must be adjacent. Otherwise, we say that σ *avoids* τ and write $\sigma \in [k]^n(\tau)$. Thus, $[k]^n(\tau)$ denotes the set of all the words in $[k]^n$ that avoid τ. Moreover, if P is a set of generalized patterns then $|[k]^n(P)|$ denotes the set all the words in $[k]^n$ that avoid each pattern from P simultaneously.

Example 1.1. *A word $\pi = a_1a_2 \ldots a_n$ avoids the pattern 13-2 if π has no subsequence $a_ia_{i+1}a_j$ with $j > i+1$ and $a_i < a_j < a_{i+1}$. Also, π avoids the pattern 121 if it has no subword $a_ia_{i+1}a_{i+2}$ such that $a_i = a_{i+2} < a_{i+1}$.*

Classical patterns are generalized patterns with all possible dashes (say, 2-1-3-4), in other words, those that place no adjacency requirements on σ. The first case of classical patterns studied was that of permutations avoiding a pattern of length 3 in $\mathfrak{S}_3$. Knuth [Kn] found that, for any $\tau \in \mathfrak{S}_3$, $|\mathfrak{S}_n(\tau)| = C_n$, the nth Catalan number. Later, Simion and Schmidt [SS] determined the number $|\mathfrak{S}_n(P)|$ of

[1] Research financed by EC's IHRP Programme, within the Research Training Network "Algebraic Combinatorics in Europe", grant HPRN-CT-2001-00272

permutations in $\mathfrak{S}_n$ simultaneously avoiding any given set of patterns $P \subseteq \mathfrak{S}_3$. Burstein [Bu] extended this to $|[k]^n(P)|$ with $P \subseteq \mathfrak{S}_3$. Burstein and Mansour [BM1] considered forbidden patterns with repeated letters. Also, Burstein and Mansour [BM2, BM3] considered forbidden generalized patterns with repeated letters.

Generalized permutation patterns were introduced by Babson and Steingrímsson [BS] with the purpose of the study of Mahonian statistics. Claesson [C] and Claesson and Mansour [CM] considered the number of permutations avoiding one or two generalized patterns with one dash. Kitaev [Ki1] examined the number of $|\mathfrak{S}_n(P)|$ of permutations in $\mathfrak{S}_n$ simultaneously avoiding any set of generalized patterns with no dashes. Besides, Kitaev [Ki2] introduced a further generalization of the generalized permutation patterns namely *partially ordered generalized patterns.*

In this paper we introduce a further generalization of the generalized patterns namely *partially ordered generalized patterns in words* (*POGPs*), which is an analogue of POGPs in permutations [Ki2]. A POGP is a generalized pattern some of whose letters are incomparable. For example, if we write $\tau = 1\text{-}1'2'$, then we mean that in an occurrence of τ in a word $\sigma \in [k]^n$ the letter corresponding to the 1 in τ can be either larger, smaller, or equal to the letters corresponding to $1'$ and $2'$, whereas the letter corresponding to $1'$ must be less than the letter corresponding to $2'$. Thus, the word $113425 \in [5]^6$ contains seven occurrence of τ, namely 113, 134 twice, 125 twice, 325, and 425.

Following [Ki2], we consider two particular classes of POGPs – *shuffle patterns* and *multi-patterns*, which allows us to give an analogue for all the main results of [Ki2] for k-ary words. A multi-pattern is of the form $\tau = \tau_0\text{-}\tau_1\text{-}\cdots\text{-}\tau_s$ and a shuffle pattern of the form $\tau = \tau_0\text{-}a_1\text{-}\tau_1\text{-}a_2\text{-}\cdots\text{-}\tau_{s-1}\text{-}a_s\text{-}\tau_s$, where for any i and j, the letter a_i is greater than any letter of τ_j and for any $i \neq j$ each letter of τ_i is incomparable with any letter of τ_j. These patterns are investigated in Sections 3 and 4.

Let $\tau = \tau_0\text{-}\tau_1\text{-}\cdots\text{-}\tau_s$ be an arbitrary multi-pattern and let $A_{\tau_i}(x;k)$ be the generating function (g.f.) for the number of words in k-letter alphabet that avoid τ_i for each i. In Theorem 4.7 we find the g.f., in terms of the $A_{\tau_i}(x;k)$, for the number of k-ary words that avoid τ. In particular, this allows us to find the g.f. for the entire *distribution* of the maximum number of non-overlapping occurrences of a pattern τ with no dashes, if we only know the g.f. for the number of k-ary words that avoid τ. Thus, in order to apply our results in what follows we need to know how many k-ary words avoid a given ordinary generalized pattern with no dashes. This question was examined, for instance, in [BM1, Sections 2 and 3], [BM2, Section 3] and [BM3, Section 3.3].

2. Definitions and Preliminaries

A *partially ordered generalized pattern* (*POGP*) is a generalized pattern where some of the letters can be incomparable. Thus, the letters of a POGP form a poset (see [Stan, Chapter 3]). If letters a and b are incomparable in a POGP τ then in an occurrence of τ in a word σ, the relative order of the letters in σ corresponding to a and b is unimportant. For the POGPs considered in this paper, one can use primes to mark comparable and incomparable letters of a POGP in the following way. If two letters, say 1 and 2, have the same number of primes $k \geq 0$, say two, then they are comparable and naturally $1'' < 2''$. Any two letters with different number of primes, say $k \geq 1$ and $\ell \geq 1$, $k \neq \ell$, are incomparable. If a letter of a POGP has no primes, that means that this letter is greater than every letter with one or more primes.

Example 2.1. *The simplest non-trivial example of a POGP that differs from the ordinary generalized patterns is* $\tau = 1'\text{-}2\text{-}1''$*, where the second letter is the greatest one and the first and the last letters* *are incomparable to each other. The word* $\sigma = 31421$ *has five occurrences of* τ*, namely* 342, 341, 142, 141*, and* 121.

Let $A_\tau(x;k) = \sum_{n\geq 0} a_\tau(n;k)x^n$ denote the *generating function (g.f.)* for the numbers $a_\tau(n;k)$ of words in $[k]^n$ avoiding the pattern τ. For $\tau = 1'\text{-}2\text{-}1''$, we have

$$A_{1'\text{-}2\text{-}1''}(x;k) = \frac{1}{(1-x)^{2k-1}} - \sum_{j=1}^{k-1} \frac{x}{(1-x)^{2j}}. \tag{2.1}$$

Indeed, if $\sigma \in [k]^n$ avoids τ, and σ contains $s > 0$ copies of the letter k, then the letters k appear as leftmost or rightmost letters of σ. If σ contains no k then $\sigma \in [k-1]^n$. So, for all $n \geq 0$, we have

$$a_\tau(n;k) = a_\tau(n;k-1) + 2a_\tau(n-1;k-1) + 3a_\tau(n-2;k-1) + \cdots + (n+1)a_\tau(0;k-1),$$

since there are $(i+1)a_\tau(n-i;k-1)$ possibilities to place i letters k into σ, for $0 \leq i \leq n$. Hence, for all $n \geq 2$,

$$a_\tau(n;k) - 2a_\tau(n-1;k) + a_\tau(n-2;k) = a_\tau(n;k-1),$$

together with $a_\tau(0,k) = 1$ and $a_\tau(1,k) = k$. Multiplying both sides of the recurrence above with x^n and summing over all $n \geq 2$, we get Equation 2.1.

Definition 2.2. *If the number of words in* $[k]^n$*, for each* n*, that avoid a POGP* τ *is equal to the number of words that avoid a POGP* ϕ*, then* τ *and* ϕ *are said to be* equivalent *and we write* $\tau \equiv \phi$.

The *reverse* $R(\sigma)$ of a word $\sigma = \sigma_1\sigma_2\ldots\sigma_n$ is the word $\sigma_n\ldots\sigma_2\sigma_1$. The *complement* $C(\sigma)$ is the word $\theta = \theta_1\theta_2\ldots\theta_n$ where $\theta_i = k+1-\sigma_i$ for all $i = 1,2,\ldots,n$. For example, if $\sigma = 123331 \in [3]^6$, then $R(\sigma) = 133321$, $C(\sigma) = 321113$, and $R(C(\sigma)) = 311123$. We call these bijections of $[k]^n$ to itself *trivial*. For example, the number of words that avoid the pattern 12-2 is the same as the number of words that avoid the patterns 2-21, 1-12, and 21-1, respectively.

Following [Ki2], it is convenient to introduce the following definition.

Definition 2.3. *Let* τ *be a generalized pattern without dashes. A word* σ quasi-avoids τ *if* σ *has exactly one occurrence of* τ *and this occurrence consists of the* $|\tau|$ *rightmost letters of* σ*, where* $|\tau|$ *denotes the number of letters in* τ.

For example, the word 5112234 quasi-avoids the pattern 1123, whereas the words 5223411 and 1123345 do not.

Proposition 2.4. *Let* τ *be a non-empty generalized pattern with no dashes. Let* $A^*_\tau(x;k)$ *denote the g.f. for the number of words in* $[k]^n$ *that quasi-avoid* τ*. Then*

$$A^*_\tau(x;k) = (kx-1)A_\tau(x;k) + 1. \tag{2.2}$$

Proof. Using the similar arguments as those in the proof of [Ki2, Proposition 4], we get that, for $n \geq 1$,

$$a^*_\tau(n;k) = ka_\tau(n-1;k) - a_\tau(n;k),$$

where $a^*_\tau(n;k)$ denotes the number of words in $[k]^n$ that quasi-avoid τ. Multiplying both sides of the last equality by x^n and summing over all natural numbers n, we get the desired result. □

Definition 2.5. *Suppose* $\{\tau_0, \tau_1, \ldots, \tau_s\}$ *is a set of generalized patterns with no dashes and*

$$\tau = \tau_0\text{-}\tau_1\text{-}\cdots\text{-}\tau_s,$$

where each letter of τ_i *is incomparable with any letter of* τ_j *whenever* $i \neq j$*. We call such POGPs* multi-patterns.

Definition 2.6. *Suppose $\{\tau_0, \tau_1, \dots, \tau_s\}$ is a set of generalized patterns with no dashes and $a_1 a_2 \dots a_s$ is a word of s letters. We define a* shuffle *pattern to be a pattern of the form*

$$\tau = \tau_0\text{-}a_1\text{-}\tau_1\text{-}a_2\text{-}\cdots\text{-}\tau_{s-1}\text{-}a_s\text{-}\tau_s,$$

where each letter of τ_i is incomparable with any letter of τ_j whenever $i \neq j$, and the letter a_i is greater than any letter of τ_j for any i and j.

For example, $1'$-2-$1''$ is a shuffle pattern, and $1'$-$1''$ is a multi-pattern. From definitions, we obtain that we can get a multi-pattern from a shuffle pattern by removing all the letters a_i.

There is a connection between multi-avoidance of the generalized patterns and the POGPs. In particular, to avoid $1'$-2-$1''$ is the same as to avoid simultaneously the patterns 1-2-1, 1-3-2, and 2-3-1. A straightforward argument leads to the following proposition.

Proposition 2.7. *For any POGP τ there exists a set T of generalized patterns such that a word σ avoids τ if and only if σ avoids all the patterns in T.*

For example, if $\tau = 1'2'$-3-$1''$, then to avoid τ is the same to avoid 5 patterns, 12-3-1, 12-3-2, 12-4-3, 13-4-2, and 23-4-1.

3. The shuffle pattern

We recall that according to Definition 2.6, a shuffle pattern is a pattern of the form

$$\tau = \tau_0\text{-}a_1\text{-}\tau_1\text{-}a_2\text{-}\cdots\text{-}\tau_{s-1}\text{-}a_s\text{-}\tau_s,$$

where $\{\tau_0, \tau_1, \dots, \tau_s\}$ is a set of generalized patterns with no dashes, $a_1 a_2 \dots a_s$ is a word of s letters, for any i and j the letter a_i is greater than any letter of τ_j and for any $i \neq j$ each letter of τ_i is incomparable with any letter of τ_j.

Let us consider the shuffle pattern $\phi = \tau$-ℓ-τ, where ℓ is the greatest letter in ϕ and each letter in the left τ is incomparable with every letter in the right τ.

Theorem 3.1. *Let ϕ be the shuffle pattern τ-ℓ-τ described above. Then for all $k \geq \ell$,*

$$A_\phi(x;k) = \frac{1}{(1 - xA_\tau(x;k-1))^2}\Big(A_\phi(x;k-1) - xA_\tau^2(x;k-1)\Big).$$

Proof. We show how to get a recurrence relation on k for $A_\phi(x;k)$, which is the g.f. for the number of words in $[k]^n(\phi)$. Suppose $\sigma \in [k]^n(\phi)$ is such that it contains exactly d copies of the letter k. If $d = 0$ then the g.f. for the number of such words is $A_\phi(x;k-1)$. Assume that $d \geq 1$. Clearly, σ can be written in the following form:

$$\sigma = \sigma_0 k \sigma_1 k \cdots k \sigma_d,$$

where σ_j is a ϕ-avoiding word on $k-1$ letters, for $j = 0, 1, \dots, d$. There are two possibilities: either σ_j avoids τ for all j, or there exists j_0 such that σ_{j_0} contains τ and for any $j \neq j_0$, the word σ_j avoids τ. In the first case, the number of such words is given by the g.f. $x^d A_\tau^{d+1}(x;k-1)$, whereas in the second case, by $(d+1)x^d A_\tau^d(x;k-1)(A_\phi(x;k-1) - A_\tau(x;k-1))$. In the last expression, the multiple $(d+1)$ is the number of ways to choose j_0, such that σ_{j_0} has an occurrence of τ, and $A_\phi(x;k-1) - A_\tau(x;k-1)$ is the g.f. for the number of words avoiding ϕ and containing τ.

Therefore,

$$A_\phi(x;k) = A_\phi(x;k-1) + \sum_{d\geq 1}(d+1)x^d A_\tau^d(x;k-1)A_\phi(x;k-1) - \sum_{d\geq 1} dx^d A_\tau^{d+1}(x;k-1),$$

equivalently,

$$A_\phi(x;k) = A_\phi(x;k-1) + A_\phi(x;k-1)\frac{2xA_\tau(x;k-1) - x^2A_\tau^2(x;k-1)}{(1-xA_\tau(x;k-1))^2} - \frac{xA_\tau^2(x;k-1)}{(1-xA_\tau(x;k-1))^2}.$$

The rest is easy to check. $\square$

Example 3.2. *Let $\phi = 1'$-2-$1''$. Here $\tau = 1$, so $A_\tau(x;k) = 1$ for all $k \geq 1$, since only the empty word avoids τ. Hence, according to Theorem 3.1, we have*

$$A_\phi(x;k) = \frac{A_\phi(x;k-1) - x}{(1-x)^2},$$

which together with $A_\phi(x;1) = \frac{1}{1-x}$ (for any n only the word $\underbrace{11\ldots1}_{n \text{ times}}$ avoids ϕ) gives Equation 2.1.

More generally, we consider a shuffle pattern of the form τ_0-ℓ-τ_1, where ℓ is the greatest element of the pattern.

Theorem 3.3. *Let ϕ be the shuffle pattern τ-ℓ-ν. Then for all $k \geq \ell$,*

$$A_\phi(x;k) = \frac{1}{(1-xA_\tau(x;k-1))(1-xA_\nu(x;k-1))}\Big(A_\phi(x;k-1) - xA_\tau(x;k-1)A_\nu(x;k-1)\Big).$$

Proof. We proceed as in the proof of Theorem 3.1. Suppose $\sigma \in [k]^n(\phi)$ is such that it contains exactly d copies of the letter k. If $d = 0$ then the g.f. for the number of such words is $A_\phi(x;k-1)$. Assume that $d \geq 1$. Clearly, σ can be written in the following form:

$$\sigma = \sigma_0 k \sigma_1 k \cdots k \sigma_d,$$

where σ_j is a ϕ-avoiding word on $k-1$ letters, for $j = 0, 1, \ldots, d$. There are two possibilities: either σ_j avoids τ for all j, or there exists j_0 such that σ_{j_0} contains τ, σ_j avoids τ for all $j = 0, 1, \ldots, j_0 - 1$ and σ_j avoids ν for any $j = j_0 + 1, \ldots, d$. In the first case, the number of such words is given by the g.f. $x^d A_\tau^{d+1}(x;k-1)$. In the second case, we have

$$x^d \sum_{j=0}^{d} A_\tau^j(x;k-1)A_\nu^{d-j}(x;k-1)(A_\phi(x;k-1) - A_\tau(x;k-1)).$$

Therefore, we get

$$\begin{aligned} A_\phi(x;k) = A_\phi(x;k-1) + A_\phi(x;k-1) &\sum_{d\geq1} x^d \sum_{j=0}^{d} A_\tau^j(x;k-1)A_\nu^{d-j}(x;k-1) \\ - &\sum_{d\geq1} x^d \sum_{j=1}^{d} A_\tau^j(x;k-1)A_\nu^{d+1-j}(x;k-1), \end{aligned}$$

equivalently,

$$A_\phi(x;k) = (A_\phi(x;k-1) - xA_\tau(x;k-1)A_\nu(x;k-1))\sum_{d\geq0} x^d \sum_{j=0}^{d} A_\tau^j(x;k-1)A_\nu^{d-j}(x;k-1).$$

Hence, using the identity $\displaystyle\sum_{n\geq0} x^n \sum_{j=0}^{n} p^j q^{n-j} = \frac{1}{(1-xp)(1-xq)}$ we get the desired result. $\square$

We now give two corollaries to Theorem 3.3.

Corollary 3.4. *Let $\phi = \tau_0\text{-}\ell\text{-}\tau_1$ be a shuffle pattern, and let $f(\phi) = f_1(\tau_0)\text{-}\ell\text{-}f_2(\tau_1)$, where f_1 and f_2 are any trivial bijections. Then $\phi \equiv f(\phi)$.*

Proof. Using Theorem 3.3, and the fact that the number of words in $[k]^n$ avoiding τ (resp. ν) and $f_1(\tau)$ (resp. $f_2(\nu)$) have the same generating functions, we get the desired result. □

Corollary 3.5. *For any shuffle pattern $\tau\text{-}\ell\text{-}\nu$, we have*

$$\tau\text{-}\ell\text{-}\nu \equiv \nu\text{-}\ell\text{-}\tau.$$

Proof. Corollary 3.4 yields that the shuffle pattern $\tau\text{-}\ell\text{-}\nu$ is equivalent to the pattern $\tau\text{-}\ell\text{-}R(\nu)$, which is equivalent to the pattern $R(\tau\text{-}\ell\text{-}R(\nu)) = \nu\text{-}\ell\text{-}R(\tau)$. Finally, we use Corollary 3.4 one more time to get the desired result. □

4. The multi-patterns

We recall that according to Definition 2.5, a multi-pattern is a pattern of the form $\tau = \tau_0\text{-}\tau_1\text{-}\cdots\text{-}\tau_s$, where $\{\tau_0, \tau_1, \dots, \tau_s\}$ is a set of generalized patterns with no dashes and each letter of τ_i is incomparable with any letter of τ_j whenever $i \neq j$.

The simplest non-trivial example of a multi-pattern is the multi-pattern $\phi = 1\text{-}1'2'$. To avoid ϕ is the same as to avoid the patterns 1-12, 1-23, 2-12, 2-13, and 3-12 simultaneously. To count the number of words in $[k]^n(1\text{-}1'2')$, we choose the leftmost letter of $\sigma \in [k]^n(1\text{-}1'2')$ in k ways, and observe that all the other letters of σ must be in a non-increasing order. Using [BM1], for all $n \geq 1$, we have

$$|[k]^n(1\text{-}1'2')| = k \cdot \binom{n+k-2}{n-1}.$$

The following theorem is an analogue to [Ki2, Theorem 21].

Theorem 4.1. *Let $\tau = \tau_0\text{-}\tau_1$ and $\phi = f_1(\tau_0)\text{-}f_2(\tau_1)$, where f_1 and f_2 are any of the trivial bijections. Then $\tau \equiv \phi$.*

Proof. First, let us prove that the pattern $\tau = \tau_0\text{-}\tau_1$ is equivalent to the pattern $\phi = \tau_0\text{-}f(\tau_1)$, where f is a trivial bijection. Suppose that $\sigma = \sigma_1\sigma_2\sigma_3 \in [k]^n$ avoids τ and $\sigma_1\sigma_2$ has exactly one occurrence of τ_0, namely σ_2. Then σ_3 must avoid τ_1, so $f(\sigma_3)$ avoids $f(\tau_1)$ and $\sigma_f = \sigma_1\sigma_2 f(\sigma_3)$ avoids ϕ. The converse is also true, if σ_f avoids ϕ then σ avoids τ. Since any word either avoids τ_0 or can be factored as above, we have a bijection between the class of words avoiding τ and the class of words avoiding ϕ. Thus $\tau \equiv \phi$.

Now, we use the considerations above as well as the properties of trivial bijections to get

$$\begin{aligned}\tau \equiv \tau_0\text{-}f_2(\tau_1) \equiv R(\tau_0\text{-}f_2(\tau_1)) \equiv R(f_2(\tau_1))\text{-}R(\tau_0) \equiv \\ \equiv R(f_2(\tau_1))\text{-}f_1(R(\tau_0)) \equiv R(f_2(\tau_1))\text{-}R(f_1(\tau_0)) \equiv f_1(\tau_0)\text{-}f_2(\tau_1).\end{aligned}$$

□

Using Theorem 4.1, we get the following corollary, which is an analogue to [Ki2, Corollary 22].

Corollary 4.2. *The multi-pattern $\tau_0\text{-}\tau_1$ is equivalent to the multi-pattern $\tau_1\text{-}\tau_0$.*

Proof. From Theorem 4.1, using the properties of the trivial bijection R, we get

$$\tau_0\text{-}\tau_1 \equiv \tau_0\text{-}R(\tau_1) \equiv R(R(\tau_1))\text{-}R(\tau_0) \equiv \tau_1\text{-}R(R(\tau_0)) \equiv \tau_1\text{-}\tau_0.$$

□

Using induction on s, Corollary 4.2, and proceeding in the way proposed in [Ki2, Theorem 23], we get

Theorem 4.3. *Suppose we have multi-patterns $\tau = \tau_0\text{-}\tau_1\text{-}\cdots\text{-}\tau_s$ and $\phi = \phi_0\text{-}\phi_1\text{-}\cdots\text{-}\phi_s$, where $\tau_0\tau_1\cdots\tau_s$ is a permutation of $\phi_0\phi_1\cdots\phi_s$. Then $\tau \equiv \phi$.*

The last theorem is an analogue to [Ki2, Theorem 23]. As a corollary to Theorem 4.3, using Theorem 4.1 and the idea of the proof of [Ki2, Corollary 24], we get the following corollary which is an analogue to [Ki2, Corollary 24].

Corollary 4.4. *Suppose we have multi-patterns $\tau = \tau_0\text{-}\tau_1\text{-}\cdots\text{-}\tau_s$ and $\phi = f_0(\tau_0)\text{-}f_1(\tau_1)\text{-}\cdots\text{-}f_s(\tau_s)$, where f_i is an arbitrary trivial bijection. Then $\tau \equiv \phi$.*

The following theorem is a good auxiliary tool for calculating the g.f. for the number of words that avoid a given POGP. For particular POGPs, it allows to reduce the problem to calculating the g.f. for the number of words that avoid another POGP which is shorter. We recall that $A^*_\tau(x;k)$ is the generating function for the number of words in $[k]^n$ that quasi-avoid the pattern τ.

Theorem 4.5. *Suppose $\tau = \tau_0\text{-}\phi$, where ϕ is an arbitrary POGP, and the letters of τ_0 are incomparable to the letters of ϕ. Then for all $k \geq 1$, we have*

$$A_\tau(x;k) = A_{\tau_0}(x;k) + A_\phi(x;k)A^*_{\tau_0}(x;k).$$

Proof. Suppose $\sigma = \sigma_1\sigma_2\sigma_3 \in [k]^n$ avoids the pattern τ, where $\sigma_1\sigma_2$ quasi-avoids the pattern τ_0, and σ_2 is the occurrence of τ_0. Clearly, σ_3 must avoid ϕ. To find $A_\tau(x;k)$, we observe that there are two possibilities: either σ avoids τ_0, or σ does not avoid τ_0. In these cases, the g.f. for the number of such words is equal to $A_{\tau_0}(x;k)$ and $A_\phi(x;k)A^*_{\tau_0}(x;k)$ respectively (the second term came from the factorization above). Thus, the statement is true. □

Corollary 4.6. *Let $\tau = \tau_1\text{-}\tau_2\text{-}\cdots\text{-}\tau_s$ be a multi-pattern such that τ_j is equal to either 12 or 21, for $j = 1, 2, \ldots, s$. Then*

$$A_\tau(x;k) = \frac{1 - \left(1 + \frac{kx-1}{(1-x)^k}\right)^s}{1-kx}.$$

Proof. According to [BM2], $A_{12}(x;k) = A_{21}(x;k) = \frac{1}{(1-x)^k}$. Using Theorem 4.5, Proposition 2.4 and induction on s, we get the desired result. □

More generally, using Theorem 4.5 and Proposition 2.4, we get the following theorem that is the basis for calculating the number of words that avoid a multi-pattern, and therefore is the main result for multi-patterns in this paper.

Theorem 4.7. *Let $\tau = \tau_1\text{-}\tau_2\text{-}\cdots\text{-}\tau_s$ be a multi-pattern. Then*

$$A_\tau(x;k) = \sum_{j=1}^{s} A_{\tau_j}(x;k) \prod_{i=1}^{j-1}((kx-1)A_{\tau_i}(x;k) + 1).$$

5. The distribution of non-overlapping generalized patterns

A descent in a word $\sigma \in [k]^n$ is an i such that $\sigma_i > \sigma_{i+1}$. Two descents i and j *overlap* if $j = i+1$. We define a new statistics, namely the *maximum number of non-overlapping descents,* or MND, in a word. For example, MND(33211) = 1 whereas MND(13211143211) = 3. One can find the distribution of this new statistic by using Corollary 4.6. This distribution is given in Example 5.2. However, we prove a more general theorem:

Theorem 5.1. *Let τ be a generalized pattern with no dashes. Then for all $k \geq 1$,*

$$\sum_{n\geq 0}\sum_{\sigma\in[k]^n} y^{N_\tau(\sigma)}x^n = \frac{A_\tau(x;k)}{1-y((kx-1)A_\tau(x;k)+1)},$$

where $N_\tau(\sigma)$ is the maximum number of non-overlapping occurrences of τ in σ.

Proof. We fix the natural number s and consider the multi-pattern $\Phi_s = \tau\text{-}\tau\text{-}\cdots\text{-}\tau$ with s copies of τ. If a word avoids Φ_s then it has at most $s-1$ non-overlapping occurrences of τ. Theorem 4.7 yields

$$A_{\Phi_s}(x;k) = \sum_{j=1}^{s} A_\tau(x;k)\prod_{i=1}^{j-1}((kx-1)A_\tau(x;k)+1).$$

So, the g.f. for the number of words that has exactly s non-overlapping occurrences of the pattern τ is given by

$$A_{\Phi_{s+1}}(x;k) - A_{\Phi_s}(x;k) = A_\tau(x;k)((kx-1)A_\tau(x;k)+1)^s.$$

Hence,

$$\sum_{n\geq 0}\sum_{\sigma\in[k]^n} y^{N_\tau(\sigma)}x^n = \sum_{s\geq 0} y^s A_\tau(x;k)((kx-1)A_\tau(x;k)+1)^s = \frac{A_\tau(x;k)}{1-y((kx-1)A_\tau(x;k)+1)}.$$

□

All of the following examples are corollaries to Theorem 5.1.

Example 5.2. *If we consider descents (the pattern* 12*) then $A_{12}(x;k) = \frac{1}{(1-x)^k}$ (see* [BM2]*), hence the distribution of* MND *is given by the formula:*

$$\sum_{n\geq 0}\sum_{\sigma\in[k]^n} y^{N_{12}(\sigma)}x^n = \frac{1}{(1-x)^k + y(1-kx-(1-x)^k)}.$$

Example 5.3. *The distribution of the maximum number of non-overlapping occurrences of the pattern* 122 *is given by the formula:*

$$\sum_{n\geq 0}\sum_{\sigma\in[k]^n} y^{N_{122}(\sigma)}x^n = \frac{x}{(1-x^2)^k + x - 1 + y(1-kx^2-(1-x^2)^k)},$$

since according to [BM3, Theorem 3.10]*, $A_{122}(x;k) = \frac{x}{(1-x^2)^k-(1-x)}$.*

Example 5.4. *If we consider the pattern* 212 *then $A_{212}(x;k) = \left(1 - x\sum_{j=0}^{k-1}\frac{1}{1+jx^2}\right)^{-1}$ (see* [BM3, Theorem 3.12]*), hence the distribution of the maximum number of non-overlapping occurrences of the* *pattern* 212 *is given by the formula:*

$$\sum_{n\geq 0}\sum_{\sigma\in[k]^n} y^{N_{212}(\sigma)}x^n = \frac{1}{1 - x\sum_{j=0}^{k-1}\frac{1}{1+jx^2} + xy\left(\sum_{j=0}^{k-1}\frac{1}{1+jx^2} - k\right)}.$$

Example 5.5. *Using* [BM3, Theorem 3.13]*, the distribution of the maximum number of non-overlapping occurrences of the pattern* 123 *is given by the formula:*

$$\sum_{n\geq 0}\sum_{\sigma\in[k]^n} y^{N_{123}(\sigma)}x^n = \frac{1}{\sum_{j=0}^{k} a_j\binom{k}{j}x^j + y\left(1 - kx - \sum_{j=0}^{k} a_j\binom{k}{j}x^j\right)},$$

where $a_{3m}=1$*,* $a_{3m+1}=-1$*, and* $a_{3m+2}=0$*, for all* $m\geq 0$*.*

References

[BS] E. Babson, E. Steingrímsson: Generalized permutation patterns and a classification of the Mahonian statistics, *Sém. Lothar. de Combin.*, B44b:18pp, (2000).

[Bu] A. Burstein, Enumeration of words with forbidden patterns, *Ph.D. thesis*, University of Pennsylvania, 1998.

[BM1] A. Burstein and T. Mansour, Words restricted by patterns with at most 2 distinct letters, *Electronic J. Combin.* **9:2** (2002), #R3.

[BM2] A. Burstein and T. Mansour, Words restricted by 3-letter generalized multipermutation patterns, preprint CO/0112281.

[BM3] A. Burstein and T. Mansour, Counting occurrences of some subword patterns, *Discrete Math. and Theor. Comp. Sci.* **6:1** (2003), 1–12.

[C] A. Claesson: Generalised Pattern Avoidance, *European J. Combin.* **22** (2001), 961-971.

[CM] A. Claesson and T. Mansour, Enumerating Permutations Avoiding a Pair of Babson-Steingrímsson Patterns, preprint CO/0107044.

[Ki1] S. Kitaev, Multi-avoidance of generalised patterns, *Discrete Math.* **260** (2003), 89-100.

[Ki2] S. Kitaev, Partially ordered generalized patterns, *Discrete Math.*, to appear.

[Kn] D. E. Knuth: *The Art of Computer Programming*, 2nd ed. Addison Wesley, Reading, MA, (1973).

[SS] R. Simion, F. Schmidt: Restricted permutations, *European J. Combin.* **6**, no. 4 (1985), 383–406.

[Stan] R. P. Stanley: *Enumerative Combinatorics*, Vol. 1, Cambridge University Press, (1997).