Linear recurrences and Chebyshev polynomials

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1 Introduction and the main result

As usual, Fibonacci polynomials $F_n(x)$, Lucas polynomials $L_n(x)$, and Pell polynomials $P_n(x)$ are defined by the second-order linear recurrence

$$t_{n+2} = at_{n+1} + bt_n, \tag{1}$$

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with given a, b, t_0, t_1 and $n \geq 0$. This sequence was introduced by Horadam [3] in 1965, and it generalizes many sequences (see [1, 4]). Examples of such sequences are Fibonacci polynomials sequence $(F_n(x))_{n\geq 0}$, Lucas polynomials sequence $(L_n(x))_{n\geq 0}$, and Pell polynomials sequence $(P_n(x))_{n\geq 0}$, when one has $a = x$, $b = t_1 = 1$, $t_0 = 0$; $a = t_1 = x$, $b = 1$, $t_0 = 2$; and $a = 2x$, $b = t_1 = 1$, $t_0 = 0$; respectively.

Chebyshev polynomials of the second kind (in this paper just Chebyshev polynomials) are defined by

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}$$

for $n \geq 0$. Evidently, $U_n(x)$ is a polynomial of degree n in x with integer coefficients. For example, $U_0(x) = 1$, $U_1(x) = 2x$, $U_2(x) = 4x^2 - 1$, and in general (see Recurrence 1 for $a = 2x$, $b = -1$, $t_0 = 1$, and $t_1 = 2x$), $U_{n+2}(x) = 2xU_{n+1}(x) - U_n(x)$. Chebyshev polynomials were invented for the needs of approximation theory, but are also widely used in various other branches of mathematics, including algebra, combinatorics, and number theory (see [5]).

Lemma 1.1 *Let $(t_n)_{n\geq 0}$ be any sequence that satisfies $t_{n+2} = 2x \cdot t_{n+1} - t_n$ with given t_0, t_1, and $n \geq 0$. Then for all $n \geq 0$,*

$$t_n = t_1 \cdot U_{n-1}(x) - t_0 \cdot U_{n-2}(x),$$

where U_m is the mth Chebyshev polynomial of the second kind.

Proof. A proof is straightforward using the relation $U_{n+2}(x) = 2xU_{n+1}(x) - U_n(x)$ and induction on n. $\square$

Let A be a tile of size 1×1 and B be a tile of size 1×2. We denote by $\mathfrak{L}_n$ the set of all *tilings* of a $1 \times n$ rectangle with tiles A and B. An element of $\mathfrak{L}_n$ can be written as a sequence of the letters A and B. For example, $\mathfrak{L}_1 = \{A\}$, $\mathfrak{L}_2 = \{AA, B\}$, and $\mathfrak{L}_3 = \{AAA, AB, BA\}$. We denote by $|\alpha|$ the number of tiles A and B in α. For example, $|AAA| = 3$ and $|AB| = 2$.

Proposition 1.2 *The number of tilings of a $1 \times n$ rectangle with tiles A and B is the Fibonacci number F_{n+1}, that is, $|\mathfrak{L}_n| = F_{n+1}$.*

Proof. The result is immediate for $n \leq 1$, so it is sufficient to show that the number of such tilings satisfies the recurrence $F_m = F_{m-1} + F_{m-2}$. To do this, we observe that there is a one-to-one correspondence between the tilings of a $1 \times (n-i)$ rectangle and the tilings of a $1 \times n$ rectangle in which the rightmost tile has length i, where $i = 1, 2$. Therefore, if we count tilings of a $1 \times n$ rectangle according to the length of the rightmost tile, we find the number of such tilings satisfies the recurrence $F_m = F_{m-1} + F_{m-2}$, as desired. $\square$

Let α be any element of $\mathfrak{L}_n$, we define β by $\beta_i = 1$ if $\alpha_i = A$; otherwise $\beta_i = 2$, and we write $\beta = \chi(\alpha)$. For example, $\chi(AAABAB) = 111212$.

Now, let us fix an integer s and a natural number q such that $q \geq 1$. Let $a_0, a_1, \ldots, a_{q-1}$, $b_0, b_1, \ldots, b_{q-1}$ be $2q$ constants and $\mathbf{a} = (a_0, a_1, \ldots, a_{q-1})$, $\mathbf{b} = (b_0, b_1, \ldots, b_{q-1})$. For any

$\alpha \in \mathfrak{L}_n$, we define $v(n;s) = v_{\mathbf{a},\mathbf{b}}(n;\alpha,q,s) = \prod_{i=1}^{|\alpha|} k(\beta_i)$ where

$$k(\beta_i) = \begin{cases} a_{(s+\beta_1+\cdots+\beta_i) \bmod q}, & \text{if } \beta_i = 1, \\ b_{(s+\beta_1+\cdots+\beta_i) \bmod q}, & \text{if } \beta_i = 2, \end{cases}$$

and $\beta = \chi(\alpha)$. For example, if $q = 3$, $a_n = n$ and $b_n = 1$ for $n = 0, 1, 2$, $s = 0$, and $\alpha = AABAB$, then we have that

$$v_{\mathbf{a},\mathbf{b}}(n;\alpha,q,s) = a_{1 \bmod 3} a_{2 \bmod 3} b_{4 \bmod 3} a_{5 \bmod 3} b_{7 \bmod 3} = a_1 a_2 b_1 a_2 b_1 = a_1 a_2^2 = 4.$$

We will be interestes in the sum of all $v_{\mathbf{a},\mathbf{b}}(n;\alpha,q,s)$ over all $\alpha \in \mathfrak{L}_n$, which is denoted by $V(n;s) = V_{\mathbf{a},\mathbf{b}}(n;q,s)$, that is, $V(n;s) = \sum_{\alpha \in \mathfrak{L}_n} v_{\mathbf{a},\mathbf{b}}(n;\alpha,q,s)$. For example, $V(1;s) = a_{(s+1) \bmod q}$ and $V(2;s) = a_{(s+1) \bmod q} a_{(s+2) \bmod q} + b_{(s+2) \bmod q}$. We extend the definition of $V(n;s)$ as $V(0;s) = 1$ and $V(n;s) = 0$ for $n < 0$. We remark that $V(n;q,s)$ can be given by a combinatorial interpretation as follows: $V(n;q,s)$ counts the number of ways to tile a boards of length n, with cells numbered $s+1$ through $s+n$, using colored tiles of size 1×1 and tiles of size 1×2. For a tile of size 1×1 on cell i, we have $a_{i \bmod q}$ color choices; and for a tile of size 1×2 on cells $i-1$ and i, we have $b_{i \bmod q}$ choices.

The main result of this paper can be formulated as follows.

Theorem 1.3 *Let $(x_n)_{n \geq 0}$ be any sequence ($x_n = x_{n;q}(\mathbf{a}, \mathbf{b})$) that satisfies*

$$x_{qn+d} = a_d \cdot x_{qn+d-1} + b_d \cdot x_{qn+d-2}, \tag{2}$$

for all $n \geq 1$, $0 \leq d \leq q-1$, *with given* $x_0, x_1, \ldots, x_{q-1}$. *Then for* $n \geq 1$, x_{qn+d} *is given by*

$$\sqrt{-J_{q;d}}^{\,n-2}\left(x_{q+d}\sqrt{-J_{q;d}}U_{n-1}(w_{q;d}) + (x_{2q+d} - I_{q;d}x_{q+d}) \cdot U_{n-2}(w_{q;d})\right),$$

for all $n \geq 1$, *where* U_m *is the* m*th Chebyshev polynomial,*

$$\begin{aligned} x_{q+d} &= V(d+1;-1)x_{q-1} + b_0V(d;0)x_{q-2} \\ x_{2q+d} &= V(q+d+1;-1)x_{q-1} + b_0V(q+d;0)x_{q-2}, \end{aligned}$$

and

$$\begin{aligned} w_{q;d} &= \frac{I_{q;d}}{2\sqrt{-J_{q;d}}}, \\ I_{q;d} &= b_{(d+1)\ mod\, q} \cdot V(q-2;d+1) + V(q;d), \\ J_{q;d} &= b_{(d+1)\ mod\, q} \cdot \Big(V(q-1;d+1)V(q-1;d) - V(q;d)V(q-2;d+1)\Big). \end{aligned} \tag{3}$$

The paper is organized as follows. In Section 2 we give a proof of Theorem 1.3, and in Section 3 we give some applications for Theorem 1.3.

2 Proofs

Throughout this section, we assume that q is a natural number ($q \geq 1$) and s is an integer. Also, let $a_0, a_1, \ldots, a_{q-1}, b_0, b_1, \ldots, b_{q-1}$ be $2q$ constants and $\mathbf{a} = (a_0, a_1, \ldots, a_{q-1})$, $\mathbf{b} = (b_0, b_1, \ldots, b_{q-1})$. We start from the following lemma.

Lemma 2.1 *Let ℓ be an integer such that $\ell \geq s+2$. Then*

$$V(\ell - s; s) = a_{\ell \bmod q} \cdot V(\ell - s - 1; s) + b_{\ell \bmod q} \cdot V(\ell - s - 2; s).$$

Proof. To verify this lemma, we observe that there is a one-to-one correspondence between the tilings of a $1 \times (\ell - s - i)$ rectangle and the tilings of a $1 \times (\ell - s)$ rectangle in which the rightmost tile has length i, where $i = 1, 2$. Hence $V(\ell - s; s) = a_{\ell \bmod q} \cdot V(\ell - s - 1; s) + b_{\ell \bmod q} \cdot V(\ell - s - 2, s)$, where the first term corresponds to the case $i = 1$ and the second one to the case $i = 2$. □

Now, let us apply this lemma to find x_{qn+d+m} in terms of x_{qn+d} and x_{qn+d-1}.

Proposition 2.2 *Let $q - 1 \geq d \geq 0$ and $n \geq 1$. Then for all $m \geq 0$,*

$$x_{qn+d+m} = V(m; d) \cdot x_{qn+d} + b_{(d+1) \bmod q} \cdot V(m-1; d+1) \cdot x_{qn+d-1}.$$

Proof. Let us prove this proposition by induction on m. Since

$$x_{qn+d+0} = 1 \cdot x_{qn+d+0} + b_{(d+1) \bmod q} \cdot 0 \cdot x_{qn+d-1},$$

$V(0;d)=1$ and $V(m;d)=0$ for $m<0$, we have that the proposition holds for $m=0$. By Recurrence 2 we get

$$x_{qn+d+1} = a_{(d+1) \bmod q} \cdot x_{qn+d} + b_{(d+1) \bmod q} \cdot x_{qn+d-1}$$

$$= V(1;d) \cdot x_{qn+d} + b_{(d+1) \bmod q} \cdot V(0;d+1) \cdot x_{qn+d-1},$$

therefore the proposition holds for $m=1$. Now, we assume that the proposition holds for $0,1,\ldots,m-1$, and prove that it holds for m. By induction hypothesis we have

$$x_{qn+d+m-2} = V(m-2;d) \cdot x_{qn+d} + b_{(d+1) \bmod q} \cdot V(m-3;d+1) \cdot x_{qn+d-1},$$

and

$$x_{qn+d+m-1} = V(m-1;d) \cdot x_{qn+d} + b_{(d+1) \bmod q} \cdot V(m-2;d+1) \cdot x_{qn+d-1},$$

hence, by Equation 2 we get

$$\begin{aligned} x_{qn+d+m} &= a_{(d+m) \bmod q} \cdot x_{qn+d+m-1} + b_{(d+m) \bmod q} \cdot x_{qn+m+d-2} \\ &= \Big(a_{(d+m) \bmod q} \cdot V(m-1;d) + b_{(d+m) \bmod q} \cdot V(m-2;d)\Big) x_{qn+d} \\ &+ b_{(d+1) \bmod q} \Big(a_{(d+m) \bmod q} \cdot V(m-2;d+1) + b_{(d+m) \bmod q} \cdot V(m-3;d+1)\Big) x_{qn+d-1}. \end{aligned}$$

Using Lemma 2.1 for $\ell = m+d$, $s=d$ and for $\ell = m+d$, $s=d+1$, we get the desired result. $\square$

Now we introduce a recurrence relation that plays the crucial role in the proof of the Main Theorem.

Proposition 2.3 *Let $q-1 \geq d \geq 0$. Then for all $n \geq 2$,*

$$x_{q(n+1)+d} = \Big(b_{(d+1)\ mod\, q} \cdot V(q-2; d+1) + V(q; d)\Big) x_{qn+d}$$
$$+ b_{(d+1)\ mod\, q} \cdot \Big(V(q-1; d+1)V(q-1; d) - V(q; d)V(q-2; d+1)\Big) x_{q(n-1)+d}.$$

Proof. Using Proposition 2.2 for $m = q-1$ we get

$$x_{q(n+1)+d-1} - b_{(d+1) \bmod q} \cdot V(q-2; d+1) \cdot x_{qn+d-1} = V(q-1; d) \cdot x_{qn+d}, \qquad (4)$$

and for $m = q$ we have

$$x_{q(n+1)+d} = V(q; d) \cdot x_{qn+d} + b_{(d+1) \bmod q} \cdot V(q-1; d+1) \cdot x_{qn+d-1}. \qquad (5)$$

Hence, Equation 4 yields

$$x_{q(n+1)+d} - b_{(d+1) \bmod q} \cdot V(q-2; d+1) \cdot x_{qn+d} =$$
$$= V(q; d)\Big(x_{qn+d} - b_{(d+1) \bmod q} \cdot V(q-2; d+1) \cdot x_{qn+d}\Big)$$
$$+ b_{(q+1) \bmod q} \cdot V(q-1; d+1)\Big(x_{qn+d-1} - b_{(d+1) \bmod q} \cdot V(q-2; d+1) \cdot x_{q(n-1)+d-1}\Big),$$

and by using Equation 4 we get the desired result. □

Proof of Theorem 1.3. Recall the definitions in 3. Now we are ready to prove the main result of this paper. Using Proposition 2.3 we have for $n \geq 2$,

$$x_{q(n+1)+d} = I_{q;d} \cdot x_{qn+d} + J_{q;d} \cdot x_{q(n-1)+d}.$$

If we define $t_n = x_{qn+d}$ for $n \geq 1$, then we get

$$t_{n+1} = I_{q;d} \cdot t_n + J_{q;d} \cdot t_{n-1},$$

therefore, by defining $(-J_{q;d})^{n/2} t'_n = t_n$ we have for $n \geq 2$,

$$t'_{n+1} = 2w_{q;d} t'_n - t'_{n-1}.$$

Let us find expressions for t'_0 and t'_1. By the recurrence for t_n we can define t_0 as $t_2 = I_{q;d} t_1 + J_{q;d} t_0$, which means that $t'_0 = t_0 = \frac{1}{J_{q;d}}(x_{2q+d} - I_{q;d} x_{q+d})$. By definitions, $t'_1 = \frac{x_{q+d}}{\sqrt{-J_{q;d}}}$. Using Proposition 2.2, we get $x_{q+d} = V(d+1;-1)x_{q-1} + b_0 V(d;0)x_{q-2}$ and $x_{2q+d} = V(q+d+1;-1)x_{q-1} + b_0 V(q+d;0)x_{q-2}$. Hence, using Lemma 1.1 we get the desired result.

3 Applications

There is a connection between the sequences which are defined by Recurrence 2, and the sequences which are define by Recurrence 1. Indeed, from Theorem 1.3 we get the following result.

Corollary 3.1 *For given x_0 and x_{-1}, and the recurrence $x_{n+2} = a_0x_{n+1} + b_0x_n$, an explicit solution for this recurrence is given by*

$$x_n = \sqrt{-b_0}^{\,n-2}\left[\sqrt{-b_0}(a_0x_0 + b_0x_{-1})U_{n-1}\left(\frac{a_0}{2\sqrt{-b_0}}\right) + b_0x_0U_{n-2}\left(\frac{a_0}{2\sqrt{-b_0}}\right)\right],$$

where U_m is the mth Chebyshev polynomial.

Proof. Using Theorem 1.3 for $q = 1$ with the parameters $d = 0$, $I_{1;0} = a_0$, $J_{1;0} = b_0$, $x_1 = a_0x_0 + b_0x_{-1}$, $x_2 = (a_0^2 + b_0)x_0 + a_0b_0x_{-1}$, and $w_{1;0} = \frac{a_0}{2\sqrt{-b_0}}$, we get the explicit solution for the recurrence $x_{n+2} = a_0x_{n+1} + b_0x_n$, as requested. □

The first interesting case is $q = 2$. Then Recurrence 2 gives

$$\begin{cases} x_{2n} &= a_0x_{2n-1} + b_0x_{2n-2} \\ x_{2n+1} &= a_1x_{2n} + b_1x_{2n-1}, \end{cases} \tag{6}$$

with given x_0 and x_1. In this case we have two possibilities: either $d = 0$ or $d = 1$. Let $d = 0$, so the parameters of the problem are given by $I_{2;0} = a_0a_1 + b_0 + b_1$, $J_{2;0} = -b_0b_1$, $w_{2;0} = \frac{a_0a_1+b_0+b_1}{2\sqrt{b_0b_1}}$, $x_2 = a_0x_1 + b_0x_0$, and $x_4 = (a_0^2a_1 + a_0b_1 + a_0b_0)x_1 + (a_0b_0a_1 + b_0^2)x_0$. Hence, Theorem 1.3 gives the following result.

Corollary 3.2 *The solution x_{2n} for Recurrence 6 is given by*

$$\sqrt{b_0b_1}^{\,n-2}\left[\sqrt{b_0b_1}(a_0x_1 + b_0x_0)U_{n-1}\left(\frac{a_0a_1 + b_0 + b_1}{2\sqrt{b_0b_1}}\right) - b_0b_1x_0U_{n-2}\left(\frac{a_0a_1 + b_0 + b_1}{2\sqrt{b_0b_1}}\right)\right],$$

where U_m is the mth Chebyshev polynomial.

Example 3.3 *If $x_0 = 0$, $x_1 = 1$, $a_0 = x$, $a_1 = xy$, and $b_0 = b_1 = 1$, then the explicit expression to x_{2n} for the Recurrence 6 is given by $xU_{n-1}\left(1 + \frac{1}{2}x^2y\right)$. Hence, by the definition it is easy to see that in the case $y = 1$, we have that the Fibonacci polynomial $F_{2n}(x)$ is given by $xU_{n-1}\left(1 + \frac{1}{2}x^2\right)$.*

If $x_0 = 2$, $x_1 = 1$, $a_0 = x$, $a_1 = xy$, and $b_0 = b_1 = 1$, then an explicit expression to x_{2n} for the Recurrence 6 is given by $(x + 2)U_{n-1}\left(1 + \frac{1}{2}x^2y\right) - 2U_{n-2}\left(1 + \frac{1}{2}x^2y\right)$. Hence, in the case $y = 1$ we have that the Lucas polynomial $L_{2n}(x)$ is given by $(x + 2)U_{n-1}\left(1 + \frac{1}{2}x^2\right) - 2U_{n-2}\left(1 + \frac{1}{2}x^2\right)$.

If $x_0 = 0$, $x_1 = 1$, $a_0 = 2x$, $a_1 = yx$, and $b_0 = b_1 = 1$, then an explicit expression to x_{2n} for the Recurrence 6 is given by $2xU_{n-1}(1 + x^2y)$. Hence, in the case $y = 2$ we have that the Pell polynomial $P_{2n}(x)$ is given by $2xU_{n-1}(1 + 2x^2)$.

Another example for Theorem 1.3 is when $q = 3$ and $d = 0$. In this case the parameters of the problem are given by $I_{3;0} = a_0a_1a_2 + b_0a_1 + b_1a_2 + a_0b_2$, $J_{3;0} = b_0b_1b_2$, $x_3 = a_0x_2 + b_0x_1$, and $x_6 - I_{3;0}x_3 = b_0b_1(x_2 - a_2x_1)$. Therefore, we get the following result.

Corollary 3.4 *The solution x_{2n} for Recurrence 2, when $q = 3$, is given by*

$$\sqrt{-b_0b_1b_2}^{\,n-2}\left(\sqrt{-b_0b_1b_2}(a_0x_2 + b_0x_1)U_{n-1}(w) + b_0b_1(x_2 - a_2x_1)U_{n-2}(w)\right),$$

for all $n \geq 1$, where $w = \frac{a_0a_1a_2 + a_0b_2 + b_0a_1 + b_1a_2}{2\sqrt{-b_0b_1b_2}}$, and U_m is the mth Chebyshev polynomial.

For example, if we are interested in solving the recurrence

$$\begin{cases} x_{3n} &= x_{3n-1} + x_{3n-2} \\ x_{3n+1} &= x_{3n} + x_{3n-1} \\ x_{3n+2} &= yx_{3n+1} + x_{3n}, \end{cases}$$

with $x_0 = 0$ and $x_1 = x_2 = 1$, then by the above corollary we get that the solution x_{3n} for this recurrence is given by

$$2i^{n-1}U_{n-1}(-i(1+y)) + i^{n-2}(1-y)U_{n-2}(-i(1+y)),$$

where $i^2 = -1$. In particular, if $y = 1$ then we have that the $(3n)$th Fibonacci number, F_{3n}, is given by $2i^{n-1}U_{n-1}(-2i)$.

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