THE PROBLEM OF THE PAWNS

Sergey Kitaev and Toufik Mansour *

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Matematik, Chalmers tekniska högskola och Göteborgs universitet, 412 96 Göteborg, Sweden
kitaev@math.chalmers.se, toufik@math.chalmers.se

Abstract

In this paper we study the number $M_{m,n}$ of ways to place nonattacking pawns on an $m \times n$ chessboard. We find an upper bound for $M_{m,n}$ and analyse its asymptotic behavior. It turns out that $\lim_{m,n\to\infty}(M_{m,n})^{1/mn}$ exists and is bounded from above by $(1+\sqrt{5})/2$. Also, we consider a lower bound for $M_{m,n}$ by reducing this problem to that of tiling an $(m+1)\times(n+1)$ board with square tiles of size 1×1 and 2×2. Moreover, we use the transfer-matrix method to implement an algorithm that allows us to get an explicit formula for $M_{m,n}$ for given m.

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1 Introduction

On an $m \times n$ chessboard, we place a number of nonattacking pawns, all of the same colour, say white. The main question here is: How many different placements are possible? A similar problem concerning placements of the

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maximum number of nonattacking kings on a $2m \times 2n$ chessboard is treated in [W]. The main result of that paper is the following theorem.

Theorem 1. *Let $f_m(n)$ denote the number of ways that mn nonattacking kings can be placed on a $2m \times 2n$ chessboard. For each $m = 1, 2, 3, \dots$ there are constants $c_m > 0$, d_m, and $0 \leq \theta_m < m + 1$ such that*

$$f_m(n) = (c_m n + d_m)(m+1)^n + O(\theta_m^n) \qquad (n \to \infty).$$

Given an $m \times n$ chessboard. We mark a square containing a pawn by 1, and a square that does not contain a pawn by 0. The placement of pawns is then completely specified by an $m \times n$ binary matrix. Moreover, to be a legal placement, the binary matrix cannot contain the following two letter words: $\begin{smallmatrix}\boxed{1} & \\ & \boxed{1}\end{smallmatrix}$ and $\begin{smallmatrix} & \boxed{1} \\ \boxed{1} & \end{smallmatrix}$ (here we use the fact that all pawns are of the same colour and thus they are allowed to attack at the same directions: either at North-West and North-East or at South-West and South-East). For example, the matrix

$$\begin{matrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{matrix}$$

corresponds to a legal placement of pawns on a 3×6 board. So, our main question can be reformulated as follows: How many binary $m \times n$ matrices simultaneously avoid the words $\begin{smallmatrix}\boxed{1} & \\ & \boxed{1}\end{smallmatrix}$ and $\begin{smallmatrix} & \boxed{1} \\ \boxed{1} & \end{smallmatrix}$? We denote the number of such matrices by $M_{m,n}$.

Studying matrices avoiding certain words, and thus studying our original problem, is interesting, for instance, from a graph theoretic point of view [CW]. In that paper, the authors considered the (vertex) independence number of the $m \times n$ grid graph using the matrices with the property that no two consecutive 1's occur in a row or a column.

In this paper, we use the transfer-matrix approach to implement an algorithm that allows us to find a formula for $M_{m,n}$ for any given m (see Sections 3 and 4). Moreover, in Section 2 we find an upper bound for $M_{m,n}$ and, in Section 3, we discuss how the tiling problem is related to finding a lower bound for $M_{m,n}$. Also, we prove that the double limit $\lim_{m,n\to\infty}(M_{m,n})^{1/mn}$ exists and is bounded from above by $(1+\sqrt{5})/2$ (see Sections 4 and 2). Finally, in Section 5, we suggest an approach to study $M_{m,n}$, which, in particular, allows to prove that $M_{2m,n}$ is a perfect square (see Theorem 7). Using this approach we obtain formulas for $M_{m,n}$, where $2 \leq m \leq 6$.

2 The upper bound for $M_{m,n}$

To obtain an upper bound for $M_{m,n}$, we determine the number $U_{m,n}$ of binary $m \times n$ matrices that avoid the word $\begin{smallmatrix}1 & \\ & 1\end{smallmatrix}$. Of course, $U_{m,n}$ counts also the number of binary $m \times n$ matrices that avoid the word $\begin{smallmatrix} & 1\\ 1 & \end{smallmatrix}$, which follows from arranging the columns of all matrices under consideration in reverse order (in particular, $\begin{smallmatrix}1 & \\ & 1\end{smallmatrix}$ is the reverse of $\begin{smallmatrix} & 1\\ 1 & \end{smallmatrix}$).

The following theorem gives a formula for the number of binary matrices that avoid the word $\begin{smallmatrix}1 & \\ & 1\end{smallmatrix}$ in terms of the Fibonacci numbers.

Theorem 2. *For any $n, m \geq 0$,*

$$U_{m,n} = \begin{cases} F_{m+1}^{n-m+1}\left(\prod_{i=0}^{m} F_i\right)^2, & \text{if } n \geq m, \\ F_{n+1}^{m-n+1}\left(\prod_{i=0}^{n} F_i\right)^2, & \text{if } n < m, \end{cases}$$

where F_i is the i-th Fibonacci number defined by $F_0 = F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$.

Proof. Let A be an $m \times n$ (0,1)-matrix that avoids the word $\begin{smallmatrix}1 & \\ & 1\end{smallmatrix}$. We change the shape of A using the following procedure. We shift the first column of A one position down with respect to the second column. In the obtained shape, we shift the first and second columns one position down with respect to the third column, and so on. After shifting with respect to the n-th column, one obtains the shape $\bar{A}$, that has the form similar to that on Figure 1.

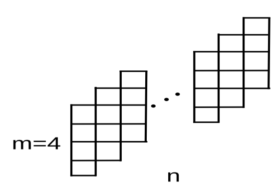

Figure 1: The shape $\bar{A}$ for $m = 4$.

It is easy to see that A avoids the word $\begin{smallmatrix}\boxed{1}&\\&\boxed{1}\end{smallmatrix}$ if and only if $\bar{A}$ avoids the word $\boxed{1}\boxed{1}$. But $\bar{A}$ avoids $\boxed{1}\boxed{1}$ if and only if each row of $\bar{A}$ avoids $\boxed{1}\boxed{1}$ (there are no additional restrictions). This is well known and is not difficult to see that the number of different binary strings of length ℓ that avoid $\boxed{1}\boxed{1}$ is given by $F_{\ell+2}$.

To find $U_{m,n}$, it remains to find out the lengths of the rows in $\bar{A}$, and since these rows are independent from each other, to multiply together the corresponding Fibonacci numbers. If $n \geq m$, $\bar{A}$ has two rows of each of the following lengths: $1, 2, \ldots, m-1$, and $n-m+1$ rows of length m. So, in this case

$$U_{m,n} = F_{m+1}^{n-m+1} \left(\prod_{i=0}^{m} F_i \right)^2 .$$

The case $n < m$ is given by changing m by n, and n by m in the considerations above. □

Let A be any binary matrix, we say that A avoids the *k-diagonal word* (see Figure 2) if there are no k consecutive 1's in any diagonal of A. Theorem 2 can be generalized to the case of avoiding the k-diagonal word. This generalization involves the k-generalized Fibonacci numbers. We do not use the generalization to proceed with the problem of the pawns, but we state it as Theorem 3 because it is interesting by its own.

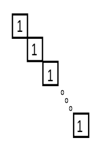

Figure 2: The k-diagonal word.

Let $F_{k,n}$ be the *n-th k-generalized Fibonacci number* defined by $F_{k,n} = 0$ for $n < 0$, $F_{k,0} = 1$, and $F_{k,n} = F_{k,n-1} + F_{k,n-2} + \cdots + F_{k,n-k}$ for $n \geq 1$ (for example, see [F, SP]).

Let $U_{m,n}(k)$ denote the number of $m \times n$ binary matrices that avoid the k-diagonal word. The following theorem can be proved using the same arguments as those in Theorem 2 and the observation that the number of different binary strings of length ℓ that avoid the word $\underbrace{11\dots1}_{k}$ is given by $F_{k,\ell+1}$ (we leave this observation as an exercise).

Theorem 3. *Let $k \geq 2$. For all $n, m \geq 0$,*

$$U_{m,n}(k) = \begin{cases} F_{k,m+1}^{n-m+1}\left(\prod_{i=0}^{m} F_{k,i}\right)^2, & \text{if } n \geq m, \\ F_{k,n+1}^{m-n+1}\left(\prod_{i=0}^{n} F_{k,i}\right)^2, & \text{if } n < m, \end{cases}$$

where $F_{k,i}$ is the i-th k-generalized Fibonacci number.

As a corollary to Theorem 2, we get an upper bound for $M_{m,n}$. Indeed, $M_{m,n} \leq U_{m,n}$ since $U_{m,n}$ deals with avoidance of $\begin{smallmatrix} \boxed{1} & \\ & \boxed{1} \end{smallmatrix}$, whereas $M_{m,n}$ deals additionaly with one more restriction, namely $\begin{smallmatrix} & \boxed{1} \\ \boxed{1} & \end{smallmatrix}$. We state this result as the following theorem.

Theorem 4. *We have*

$$M_{m,n} \leq \begin{cases} F_{m+1}^{n-m+1}\left(\prod_{i=0}^{m} F_{i}\right)^2, & \text{if } n \geq m, \\ F_{n+1}^{m-n+1}\left(\prod_{i=0}^{n} F_{i}\right)^2, & \text{if } n < m, \end{cases}$$

where F_i is the i-th Fibonacci number.

The upper bound for $M_{m,n}$ involves the product of the first nonzero Fibonacci numbers. It is known [SP, A003266] that an asymptotic for the product of the first n nonzero Fibonacci numbers is given by

$$\frac{c}{\sqrt{5}^{n-1}}\left(\frac{1+\sqrt{5}}{2}\right)^{\frac{n(n-1)}{2}}, \tag{1}$$

where $c = \prod_{j\geq 1}\left(1 - \left(\frac{\sqrt{5}-3}{2}\right)^j\right) = 1.2267420107203532444176302\ldots$. This result and Theorem 4 give the following theorem.

Theorem 5. *We have*

$$\lim_{n,m\to\infty} (M_{m,n})^{\frac{1}{mn}} \leq \frac{1+\sqrt{5}}{2}.$$

Proof. The existence of the limit $\lim_{n,m\to\infty}(M_{m,n})^{\frac{1}{mn}}$ is proved in Theorem 6. Using Theorem 4, it is enough to prove that

$$\lim_{n,m\to\infty} (U_{m,n})^{\frac{1}{mn}} = \frac{1+\sqrt{5}}{2}. \tag{2}$$

For given two functions $f(n)$ and $g(n)$, we define $f(n) \sim g(n)$ if $\lim\limits_{n\to\infty} \frac{f(n)}{g(n)} = 1$.

Suppose $n \geq m$. By (1) we have

$$\left(\prod_{i=0}^{m} F_i\right)^2 \sim \frac{c^2}{\sqrt{5}^{2m-2}} \left(\frac{1+\sqrt{5}}{2}\right)^{m^2-m},$$

and using the formula for the Fibonacci numbers, namely

$$F_m = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2}\right)^{m+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{m+1} \right),$$

we obtain that

$$U_{m,n} \sim \frac{c^2}{\sqrt{5}^{n+m-1}} \left(\frac{1+\sqrt{5}}{2}\right)^{nm+2n-2m+2}.$$

This formula holds for the case $n < m$, by replacing m, n by n, m in the considerations above. Hence, (2) holds. □

3 Tiling rectangles and a lower bound for $M_{m,n}$

Let $L_{m,n}$ denote the number of $m \times n$ binary matrices that simultaneously avoid the words $\begin{smallmatrix}1 & \\ & 1\end{smallmatrix}$, $\begin{smallmatrix} & 1\\ 1 & \end{smallmatrix}$, $\begin{smallmatrix}1\\ 1\end{smallmatrix}$ and $\begin{smallmatrix}1 & 1\end{smallmatrix}$. Clearly, $L_{m,n} \leq M_{m,n}$, since when we deal with $L_{m,n}$ we have more restrictions than when we consider $M_{m,n}$.

Thus, we are interested in finding the numbers $L_{m,n}$, that give us a lower bound for $M_{m,n}$. In this section we show that $L_{m,n}$, in fact, gives the number of tilings of an $(m+1)\times(n+1)$ area with square tiles of size 1×1 and 2×2 which was studied in [H] and [CH]. So, the number of the tilings is equal to the number of $m\times n$ binary matrices that avoid the words $\begin{smallmatrix}1&\\&1\end{smallmatrix}$, $\begin{smallmatrix}&1\\1&\end{smallmatrix}$, $\begin{smallmatrix}1\\1\end{smallmatrix}$ and $\begin{smallmatrix}1&1\end{smallmatrix}$. A bijection θ between these two combinatorial objects is given by the following.

Let A be an $m\times n$ matrix that avoids the words $\begin{smallmatrix}1&\\&1\end{smallmatrix}$, $\begin{smallmatrix}&1\\1&\end{smallmatrix}$, $\begin{smallmatrix}1\\1\end{smallmatrix}$ and $\begin{smallmatrix}1&1\end{smallmatrix}$. We make from A an $(m+1)\times(n+1)$ matrix $\bar{A}$ by adjoin an additional $m\times 1$ column consisting of 0's from the right side, and an additional $1\times(n+1)$ row, also having only 0's, from below. Now, once we meet an occurrence of 1 in $\bar{A}$, we place a 2×2 tile in such way, that the 1 appears in the top-left corner of the tile. After considering all 1's and placing corresponding 2×2 tiles, we fill in the uncovered squares of $\bar{A}$ by 1×1 tiles. The fact that A avoids $\begin{smallmatrix}1&\\&1\end{smallmatrix}$, $\begin{smallmatrix}&1\\1&\end{smallmatrix}$, $\begin{smallmatrix}1\\1\end{smallmatrix}$ and $\begin{smallmatrix}1&1\end{smallmatrix}$ guarantees that covering in the way proposed by us is non-overlapping, and thus we get a tiling of an $(m+1)\times(n+1)$ board.

Conversely, for any given tiling with square tiles of size 1×1 and 2×2, we can place 1 in the top-left corner of any 2×2 tile, 0's in the other squares, and remove the rightmost column and the bottom row. Obviously, we get an $m\times n$ binary matrix that avoids the words $\begin{smallmatrix}1&\\&1\end{smallmatrix}$, $\begin{smallmatrix}&1\\1&\end{smallmatrix}$, $\begin{smallmatrix}1\\1\end{smallmatrix}$ and $\begin{smallmatrix}1&1\end{smallmatrix}$.

Figure 3 shows how the bijection θ works in the case of a 4×5 matrix.

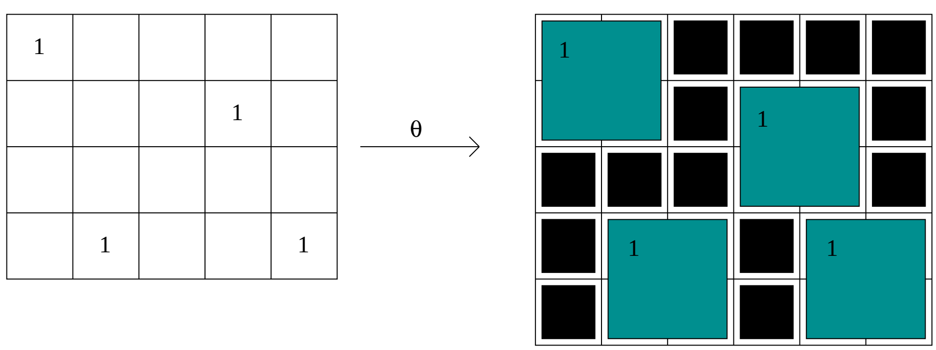

Figure 3: The bijection θ.

Unfortunately, we cannot get much use of the papers [H] and [CH], since there, for our purpose, one has explicit formulas only for $m = 2, 3$, and for $m = 4, 5$ one has recursive formulas only. That means that from that source, we have an information about $L_{m,n}$, where $1 \leq m \leq 4$.

Remark 1. *If we use the transfer-matrix approach* (see [M, pawns-kings]) *for finding the formula for $L_{m,n}$, where m is given, then we get the following: $L_{1,n} = F_{n+1}$, $L_{2,n} = \frac{1}{3}(2^{n+2} - (-1)^n)$, and*

$$L_{3,n} = a\left(\tfrac{2}{3} + \tfrac{2\sqrt{13}}{3}\cos\beta\right)^n + b\left(\tfrac{2}{3} - \tfrac{2\sqrt{39}}{3}\sin\beta - \tfrac{2\sqrt{13}}{3}\cos\beta\right)^n + c\left(\tfrac{2}{3} + \tfrac{2\sqrt{39}}{3}\sin\beta - \tfrac{2\sqrt{13}}{3}\cos\beta\right)^n,$$

where $\beta = \frac{1}{3}\arctan\left(\frac{3}{8}\sqrt{237}\right)$, $a \approx 1.51212496094$, $b \approx -0.542960193686$, and $c \approx 0.0308352327442$.

4 The transfer-matrix method

We use the transfer-matrix method, in a manner that is similar to way it was used in [W].

For m, n fixed, we can think of constructing the $m \times n$ binary matrices avoiding the words $\begin{smallmatrix}1 & \\ & 1\end{smallmatrix}$ and $\begin{smallmatrix} & 1\\ 1 & \end{smallmatrix}$ by gluing together columns that are chosen from the collection of possible columns, making sure that when we glue an additional column onto the right-hand edge of the structure, the new column does not come into conflict with the previous right-hand column. The collection of possible columns C_m is the set of all m-vectors v of 0's and 1's. Clearly, $|C_m| = 2^m$.

The condition that vectors v, w in C_m are possible consecutive pair of columns in a matrix avoiding $\begin{smallmatrix}1 & \\ & 1\end{smallmatrix}$ and $\begin{smallmatrix} & 1\\ 1 & \end{smallmatrix}$ is simply that $v_i w_{i+1} = 0$ and $v_{i+1} w_i = 0$ for all i, $1 \leq i \leq m-1$. We say that such v and w are *cross-orthogonal.*

Thus, all possible matrices under consideration are obtained by beginning with some vector of C_m, and in general, having arrived at some sequence of vectors of C_m, adjoin any vector of C_m that is cross-orthogonal to the last one previously chosen until n vectors have been selected.

We define a matrix $T = T_m$, the transfer matrix of the problem, as follows. T is an $2^m \times 2^m$ symmetric matrix of 0's and 1's whose rows and columns are indexed by vectors of C_m. The entry of T in position (v, w) is 1 if the vectors v, w are cross-orthogonal, and 0 otherwise. T depends only on m, not on n.

Let $M_{m,n}(u)$ denote the number of $m \times n$ binary matrices avoiding the words $\begin{smallmatrix}1 & \\ & 1\end{smallmatrix}$ and $\begin{smallmatrix} & 1\\ 1 & \end{smallmatrix}$ whose rightmost column vector is u. Then, clearly, we have

$$M_{m,n+1}(v) = \sum_{u \in C_m} M_{m,n}(u) T_{u,v} \qquad (n \geq 0; v \in C_m),$$

or, in matrix-vector notation, $M_{n+1} = TM_n$, with $M_0 = \mathbf{1}$ the vector of length 2^m whose entries are all 1's. It follows that $M_n = T^n \cdot \mathbf{1}$, for all $n \geq 0$. The number of matrices $M_{m,n}$ is the sum of the entries of the vector M_n. Thus, if $\mathbf{1'}$ denote the row of length 2^m whose entries are all 1's, we have

$$M_{m,n} = \mathbf{1'} \cdot T^n \cdot \mathbf{1},$$

i.e., $M_{m,n}$ is the sum of all of the entries of the matrix T^n.

Example 1. *The transfer-matrices T_2 and T_3* (see [M, pawns]) *are given, for instance, by*

$$T_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \textit{and} \quad T_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since T has nonnegative entries, its dominant eigenvector cannot be orthogonal to $\mathbf{1}$, and so we have at once that $\lim_{n\to\infty}(M_{m,n})^{\frac{1}{n}}$ exists for each m, and is equal to α_m, the largest eigenvalue of the transfer-matrix T (real and symmetric matrix). It follows that

$$\liminf_{m} (\alpha_m)^{\frac{1}{m}} = \liminf_{m,n} (M_{m,n})^{\frac{1}{mn}} \leq \limsup_{m,n} (M_{m,n})^{\frac{1}{mn}} = \limsup_{m} (\alpha_m)^{\frac{1}{m}}. \quad (3)$$

Theorem 6. *The limit $\lim_{m,n\to\infty}(M_{m,n})^{\frac{1}{mn}}$ exists.*

Proof. By the fact that T is symmetric and real matrix together with using the maximum principle we get, for any $q \geq 1$,

$$\frac{(\mathbf{1}, (T_m)^q \cdot \mathbf{1})}{(\mathbf{1}, \mathbf{1})} \leq (\alpha_m)^q.$$

Since $M_{m,q} = M_{q,m}$ by the definitions, we have $(\mathbf{1}, (T_m)^q \cdot \mathbf{1}) = (\mathbf{1}, (T_q)^m \cdot \mathbf{1})$. Hence,

$$\left(\frac{(\mathbf{1}, (T_q)^m \cdot \mathbf{1})}{(\mathbf{1}, \mathbf{1})} \right)^{\frac{1}{m}} \leq (\alpha_m)^{\frac{q}{m}}.$$

Taking the $\liminf_m$ of both sides of the inequality above, together with using the fact that $|C_m| = 2^m$, we have $\frac{\alpha_q}{2} \leq \left(\liminf_m (\alpha_m)^{\frac{1}{m}} \right)^q$, which implies

$$\limsup_q (\alpha_q)^{\frac{1}{q}} \leq \liminf_m (\alpha_m)^{\frac{1}{m}}.$$

Using (3) we get the desired result. □

Using the transfer-matrix approach one can obtain an explicit formula for $M_{m,n}$, where $m \geq 1$ is given. We implemented an algorithm for finding the transfer-matrix T_m in Maple (see [M, pawns]). This algorithm yields an explicit formula for $M_{m,n}$, where $1 \leq m \leq 3$ (see Table 1). Moreover, it finds the maximum eigenvalue of T_m for given m.

m	$M_{m,n}$
1	2^n
2	$\frac{7}{10}(\eta_1^{2n} + \eta_2^{2n}) + \frac{3\sqrt{5}}{10}(\eta_1^{2n} - \eta_2^{2n}) - \frac{2}{5}(-1)^n$, where $\eta_1 = \frac{1}{2}(1+\sqrt{5})$ and $\eta_2 = \frac{1}{2}(1-\sqrt{5})$
3	$\frac{1}{13}\left(\eta_1^{n+2} + \eta_2^{n+2}\right) + \frac{\sqrt{3}^{n+1}}{13}\left(4 - \sqrt{3} - (4+\sqrt{3})(-1)^n\right)$, where $\eta_1 = \frac{1}{2}(5+\sqrt{13})$ and $\eta_2 = \frac{1}{2}(5-\sqrt{13})$

Table 1: Explicit formula for $M_{m,n}$ where $m = 1, 2, 3$.

For example, the maximum eigenvalue of T_m is 2, $\left(\frac{1+\sqrt{5}}{2}\right)^2$, $\frac{5+\sqrt{13}}{2}$ and $\frac{8}{3} + \frac{4}{3}\sqrt{7}\cos\left(\frac{1}{3}\arctan\left(\frac{3}{67}\sqrt{111}\right)\right)$, for $m = 1, 2, 3, 4$; respectively.

Remark 2. *In the case of $m = 4$, the eigenvalues of T are given by*

$$\begin{aligned}
\lambda_1 &= \tfrac{2}{3} - \tfrac{4}{3}\cos\left(\tfrac{1}{3}\pi - \beta\right) - \tfrac{4}{3}\sqrt{3}\sin\left(\tfrac{1}{3}\pi - \beta\right), \\
\lambda_2 &= \tfrac{2}{3} - \tfrac{4}{3}\cos\left(\tfrac{1}{3}\pi - \beta\right) + \tfrac{4}{3}\sqrt{3}\sin\left(\tfrac{1}{3}\pi - \beta\right), \\
\lambda_3 &= \tfrac{8}{3} - \tfrac{2}{3}\sqrt{7}\cos\gamma - \tfrac{2}{3}\sqrt{21}\sin\gamma, \\
\lambda_4 &= \tfrac{8}{3} - \tfrac{2}{3}\sqrt{7}\cos\gamma + \tfrac{2}{3}\sqrt{21}\sin\gamma, \\
\lambda_5 &= -\tfrac{2}{3} - \tfrac{4}{3}\cos\beta - \tfrac{4}{3}\sqrt{3}\sin\beta, \\
\lambda_6 &= -\tfrac{2}{3} - \tfrac{4}{3}\cos\beta + \tfrac{4}{3}\sqrt{3}\sin\beta, \\
\lambda_7 &= \tfrac{2}{3} + \tfrac{8}{3}\cos\left(\tfrac{1}{3}\pi - \beta\right), \\
\lambda_8 &= -\tfrac{2}{3} + \tfrac{8}{3}\cos\left(\tfrac{1}{3}\pi - \beta\right), \\
\lambda_9 &= \tfrac{8}{3} + \tfrac{4}{3}\sqrt{7}\cos\gamma,
\end{aligned}$$

where $\beta = \frac{1}{3}\arctan\left(\frac{3}{5}\sqrt{111}\right)$ *and* $\gamma = \frac{1}{3}\arctan\left(\frac{3}{67}\sqrt{(111)}\right)$.

5 Formulas for $M_{m,n}$

In this section we suggest another approach to study $M_{m,n}$. In particular, we obtain formulas for $M_{m,n}$, where $2 \leq m \leq 6$ (the cases $m = 2, 3$ already appear in Table 1). We show how to use the following simple observation in order to investigate $M_{m,n}$.

Observation 1. *A pawn placed on a square of a chessboard cannot attack a square of the different colour.*

According to Observation 1, $M_{m,n} = B_{m,n} \cdot W_{m,n}$, where $B_{m,n}$ (resp. $W_{m,n}$) is the number of ways to place nonattacking pawns on the black (resp. white) squares of an $m \times n$ chessboard. Thus, the original problem of finding $M_{m,n}$ can be reduced to considering independently two shapes: that consisting of all the black squares, and the shape consisting of all the white squares. We use this idea in the proofs of the following theorem and propositions.

Theorem 7. *We have that $M_{2m,n} = a^2$ for some natural number a, that is $M_{2m,n}$ is a perfect square.*

Proof. Using the discussion right above this theorem, it is enough to prove that $B_{2m,n} = W_{2m,n}$. Indeed, on a $2m \times n$ chessboard, the number of black squares is the same as that of white squares. Moreover, if we consider the shape that, say, the white squares form, reverse it horizontally (that is, draw the rows in reverse order), then we get exactly the same shape that the black squares form. Also, it is easy to see that a placement of pawns before the reversion is legal if and only if it is legal after the reversion. Thus, we have $B_{2m,n} = W_{2m,n}$. □

Proposition 1. *We have*

$$M_{2,n} = (F_{n+2})^2,$$

where F_n is the n-th Fibonacci number defined by $F_0 = F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$.

Proof. Let us draw the black squares of a $2 \times n$ chessboard in one $1 \times n$ row, in the order we meet these squares in the chessboard by going from left to right. Obviously, we have a legal placement of pawns on the chessboard if and only if we have no two consecutive pawns in the row, or in terms of matrices and word avoidance, the row avoids the word $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}$. The number of different legal rows is given by the $(n+2)$-nd Fibonacci number, that is $B_{2,n} = F_{n+2}$.

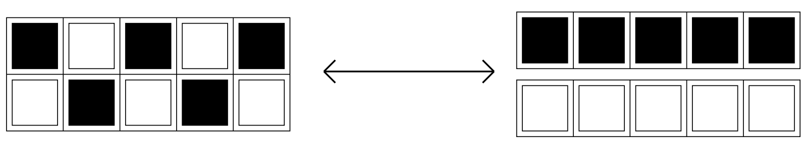

Figure 4: Finding $M_{2,5}$.

Independently, we can make the same considerations with the white squares on the chessboard to get $W_{2,n} = F_{n+2}$. Thus, $M_{2,n} = (F_{n+2})^2$. For instance, Figure 4 shows that for finding $M_{2,5}$ one can consider two rows of length 5. □

Proposition 2. *For all $n \geq 0$, $M_{3,2n+1} = (4t_n - 3t_{n-1})(2t_n - 3t_{n-1})$ and $M_{3,2n} = t_n^2$, where*

$$t_n = \frac{1}{\sqrt{13}}\left(\left(\frac{5+\sqrt{13}}{2}\right)^{n+1} - \left(\frac{5-\sqrt{13}}{2}\right)^{n+1}\right).$$

Proof. Let a_n (resp. b_n) denote the number of legal placements of pawns in the first (resp. second) shape on Figure 5 defined by black squares (there are n columns in each shape). According to Observation 1, one has $M_{3,n} = a_n b_n$. Let us find a_n and b_n.

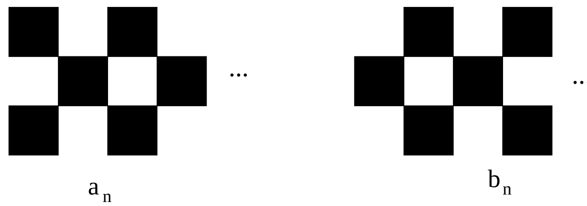

Figure 5: The shapes under consideration.

We consider the first shape. There are two black squares in the first column. Depending on whether or not these squares have pawns, we have four possibilities. As before, we use 1 for a square having a pawn, and 0 otherwise. Thus, the first column of the shape is either 00, or 01, or 10, or 11 when reading from top to bottom. In the first case, the first column of the shape does not affect the rest of the shape, and therefore can be removed. So, in the first case the number of placements of pawns is b_{n-1}. In the second, third and fouth cases, the black square in the second column must contain no pawn, that is 0, in order to have a legal placement. This 0 does not affect what follows to the right of it, and thus two first columns of the shape can be removed. So, the second, third and fouth cases give $3a_{n-2}$ placements of pawns. Thus, $a_n = 3a_{n-2} + b_{n-1}$. Similarly, one can consider the second shape to get $b_n = a_{n-1} + b_{n-2}$. Solving the equations for a_n and b_n, we have

$$a_{2n} = t_n, \; a_{2n+1} = 4t_n - 3t_{n-1}, \; b_{2n} = t_n, \text{ and } b_{2n+1} = 2t_n - 3t_{n-1}.$$

This gives the desired result. □

Remark 3. *If $a(x)$ and $b(x)$ denote the generating functions for the numbers a_n and b_n respectively in the proof of Proposition 2, then*

$$a(x) = \frac{1 + 4x - 3x^3}{1 - 5x^2 + 3x^4} \quad \text{and} \quad b(x) = \frac{1 + 2x - 3x^3}{1 - 5x^2 + 3x^4}.$$

Proposition 3. *We have that* $M_{4,n} = \alpha_n^2$*, where the generating function for the numbers* α_n *is given by*

$$\frac{1+2x-2x^2}{1-2x-2x^2+2x^3}.$$

Proof. Let α_n (resp. β_n, γ_n, δ_n) denote the number of legal placements of pawns in the first (resp. second, third, fouth) shape on Figure 6 defined by black squares (there are n columns in each shape). As in the proof of Theorem 7, using horisontal reverse of rows, it is easy to see that $\alpha_n = \beta_n$ and $\gamma_n = \delta_n$. Now, according to Observation 1, one has $M_{4,n} = \alpha_n\beta_n = (\alpha_n)^2$. Let us find α_n.

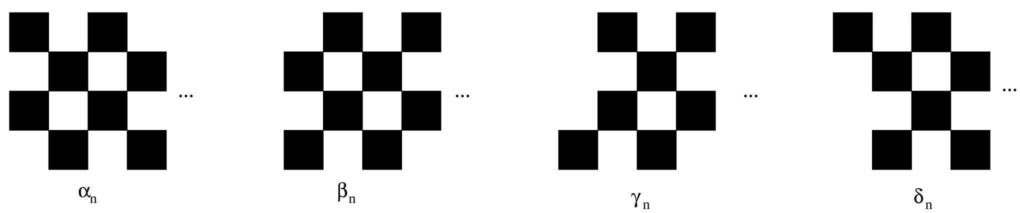

Figure 6: The shapes under consideration.

We proceed in the same way as we do in Proposition 2. If the first column of the first shape is 00 when reading the content of the black squares from top to bottom, we can remove this column since it does not affect the rest of the shape. So, in this case the number of legal placements of pawns is $\beta_{n-1} = \alpha_{n-1}$. If instead of 00 we have 01 or 11, the content of the black squares in the second column must be 00, in which case we can remove the first two columns since they do not affect the rest of the shape. So, in this case we have $2\alpha_{n-2}$ placements. The case left is when the first column is 10. In this case the top element of the second column must be 0, and we have no information concerning the second element in this column. Thus, in this case we have γ_{n-1} replacements. Therefore,

$$\alpha_n = \alpha_{n-1} + 2\alpha_{n-2} + \gamma_{n-1}. \quad (4)$$

Now, to proceed further with finding α_n, we need to find γ_n. If the element in the first column of the third shape is 0, then we can remove this element, which gives $\beta_{n-1} = \alpha_{n-1}$ replacements of pawns. If this element is 1, then the bottom element in the second column must be 0, which obviously gives $\delta_{n-1} = \gamma_{n-1}$ replacements. Thus,

$$\gamma_n = \gamma_{n-1} + \alpha_{n-1}. \tag{5}$$

Now, from Equations (4) and (5) we have

$$\alpha_n = 2\alpha_{n-1} + 2\alpha_{n-2} - 2\alpha_{n-3},$$

which gives the desired result. □

In the way similar to that Propositions 2 and 3 are proved, on can prove the following two propositions, which we state without proof.

Proposition 4. *For all $n \geq 0$, $M_{5,n} = \alpha_n\beta_n$, where the generating functions for the numbers α_n and β_n are given by*

$$\frac{1+7x-4x^2-7x^3+5x^4}{(1+x)(1-2x-6x^2+10x^3-4x^4)}$$

and

$$\frac{1+3x+x^2-5x^3+4x^4}{(1+x)(1-2x-6x^2+10x^3-4x^4)},$$

respectively.

Proposition 5. *We have that $M_{6,n} = \alpha_n^2$, where the generating function for the numbers α_n is given by*

$$\frac{1+5x-9x^2-5x^3+6x^4}{1-3x-6x^2+11x^3+5x^4-6x^5}.$$

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