# INEQUALITIES FOR WEIERSTRASS PRODUCTS

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Abstract

The aim of this note is to find new inequalities for the Weierstrass products $\prod_{i=1}^{n}(1+x_i)$ and $\prod_{i=1}^{n}(1-x_i)$, which are an analogues of Alzer's inequalities [A].

## 1. Introduction

Several authors interested in Weierstrass products $\prod_{i=1}^{n}(1+x_i)$ and $\prod_{i=1}^{n}(1-x_i)$. In [B, Pages 104–105] proved

$$1-\sum_{i=1}^{n}x_i \le \prod_{i=1}^{n}(1-x_i) \le \left(1+\sum_{i=1}^{n}x_i\right)^{-1},$$

and

$$1+\sum_{i=1}^{n}x_i \le \prod_{i=1}^{n}(1+x_i) \le \left(1-\sum_{i=1}^{n}x_i\right)^{-1},$$

where $x_i \in [0,1]$, $i=1,2,\dots,n$, and in the second inequality assumed that $\sum_{i=1}^{n} x_i < 1$. In the last years these products attacked by many other inequalities which have been published. For example, Klamkin and Newman [KN] discovered several extensions and new inequalities:

$$(n+1)^n\prod_{i=1}^{n}x_i \le \prod_{i=1}^{n}(1+x_i), \quad (n-1)^n\prod_{i=1}^{n}x_i \le \prod_{i=1}^{n}(1-x_i),$$

and in [K] presented another inequality:

$$(n+1)^n\prod_{i=1}^{n}(1-x_i) \le (n-1)^n\prod_{i=1}^{n}(1+x_i),$$

where $x_i \in [0,1]$, $i=1,2,\dots,n$, and $\sum_{i=1}^{n} x_i = 1$. An another example, the Ky Fan's (see [B, Page 5]) inequality:

$$\left(\prod_{i=1}^{n}\frac{x_i}{1-x_i}\right)^{1/n} \le \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n}(1-x_i)}, \tag{1.1}$$

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where $x_i \in \left(0, \frac{1}{2}\right]$, $i = 1, 2, \ldots, n$. In this paper, we are interesting in the Alzer's inequalities [A] of Weierstrass Products. Here, we present a generalization for the Alzer's inequalities by using the same arguments proof as in Alzer's paper [A]. In Section 2 we establish new inequalities for Weierstrass products which are an analogues for these inequalities in [A]. Finally, we apply the applications of [A] to get new inequalities for trigonometric and integral sums and products.

## 2. Inequalities for Weierstrass products

Let us start by proving two lemmas which are an analogues of the two lemmas in [A].

**Lemma 2.1.** *For all real $x_i \in [0, 1]$, $i = 1, 2, \ldots, n$, and $a \in [0, 1]$, we have*

$$\prod_{i=1}^{n} x_i \left[1 + a \sum_{i=1}^{n} (1 - x_i)\right] + \prod_{i=1}^{n} (1 - x_i) \left[1 + a \sum_{i=1}^{n} x_i\right] \leq \frac{1}{2} a + 1. \tag{2.1}$$

*Proof.* Using the same arguments as in the proof of [A, Lemma 1] we get as follows. Let

$$S(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i \left[1 + a \sum_{i=1}^{n} (1 - x_i)\right] + \prod_{i=1}^{n} (1 - x_i) \left[1 + a \sum_{i=1}^{n} x_i\right],$$

since $a, x_i \in [0, 1]$ for all $i$, we arrive at

$$\begin{aligned} S(x_1, \ldots, x_n) - S(x_1, \ldots, x_{n-1}) = & -a(1 - x_n) \prod_{i=1}^{n-1} x_i \sum_{i=1}^{n} (1 - x_i) - (1 - a)(1 - x_n) \prod_{i=1}^{n-1} x_i \\ & -a x_n \prod_{i=1}^{n-1} (1 - x_i) \sum_{i=1}^{n} x_i - (1 - a) x_n \prod_{i=1}^{n-1} (1 - x_i) \leq 0. \end{aligned}$$

Hence, inductively on $n$ we have

$$S(x_1, \ldots, x_n) \leq S(x_1) = -2a \left(x_1 - \frac{1}{2}\right)^2 + 1 + \frac{1}{2} a \leq \frac{1}{2} a + 1,$$

as requested. □

The next lemma is a counterpart of Ky Fan's Inequality 1.1 and analogues of [A, Lemma 2].

**Lemma 2.2.** *For all real $x_i \in \left(0, \frac{1}{2}\right]$, $i = 1, 2, \ldots, n$, and $a \geq 0$, we have*

$$\frac{\prod_{i=1}^{n} x_i}{\prod_{i=1}^{n} (1 - x_i)} \leq \frac{1 + a \sum_{i=1}^{n} x_i}{1 + a \sum_{i=1}^{n} (1 - x_i)}, \tag{2.2}$$

*with equality if and only if $x_i = \frac{1}{2}$ for all $i = 1, 2, \ldots, n$.*

*Proof.* To this theorem we present two different proofs: by induction argument, and by Ky Fan's Inequality 1.1.

(1) It is easy to see that the equality holds in 2.2 for $x_i = \frac{1}{2}$ where $i = 1, 2, \ldots, n$. Now, we prove the lemma by induction on $n$. For $n = 1$, Inequality 2.2 yields

$$\frac{x_1}{1 - x_1} \leq \frac{1 + a x_1}{1 + a(1 - x_1)},$$

equivalently, $x_1 \leq \frac{1}{2}$ and the sign of equality holds if and only if $x_1 = \frac{1}{2}$.
Now, let us suppose that

$$0 < x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1} \leq \frac{1}{2}, \quad x_1 < x_{n+1},$$

and further we assume that Inequality 2.2 is true for $n$. Therefore, by induction hypothesis we get

$$\frac{\prod_{i=1}^{n+1} x_i}{\prod_{i=1}^{n+1}(1-x_i)} \leq \frac{1 + a\sum_{i=1}^{n} x_i}{1 + a\sum_{i=1}^{n}(1-x_i)} \cdot \frac{x_{n+1}}{1-x_{n+1}}.$$

If putting $p = a\sum_{i=1}^{n} x_i$, $q = a\sum_{i=1}^{n}(1-x_i)$, and $x = x_{n+1}$, then we have to prove that

$$\frac{x(1+p)}{(1-x)(1+q)} < \frac{1+p+ax}{1+q+a(1-x)},$$

equivalently,

$$a(q-p)x^2 + x\Big(2(1+p)(1+q) - a(q-p)\Big) - (1+p)(1+q) < 0.$$

Since, $p = q$ if and only if $x_1 = x_2 = \ldots = x_n = \frac{1}{2}$ we have $p < q$ and since $0 \leq x \leq \frac{1}{2}$ we obtain

$$a(q-p)x^2 + x\Big(2(1+p)(1+q) - a(q-p)\Big) - (1+p)(1+q) < -\frac{a(q-p)}{4} < 0.$$

Thus we have the desired result.

(2) Since $x_i \leq \frac{1}{2}$ for all $i$, we have that $\frac{x_i}{1-x_i} \leq 1$. Therefore, by Inequality 1.1 we have

$$\prod_{i=1}^{n} \frac{x_i}{1-x_i} \leq \left[\prod_{i=1}^{n} \frac{x_i}{1-x_i}\right]^{1/n} \leq \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n}(1-x_i)} = \frac{a\sum_{i=1}^{n} x_i}{a\sum_{i=1}^{n}(1-x_i)}.$$

Hence, because of $\frac{x}{y} \leq 1$ if and only if $\frac{x}{y} \leq \frac{1+x}{1+y}$ where $x, y > 0$, and since $a\sum_{i=1}^{n} x_i \leq a\sum_{i=1}^{n}(1-x_i)$, we arrive at

$$\prod_{i=1}^{n} \frac{x_i}{1-x_i} \leq \frac{1 + a\sum_{i=1}^{n} x_i}{1 + a\sum_{i=1}^{n}(1-x_i)}.$$

□

For example, Theorem 2.2, for $a = 2$, yields [A, Lemma 2]. An another example, Theorem 2.2, for $a \to \infty$, yields

$$\frac{\prod_{i=1}^{n} x_i}{\prod_{i=1}^{n}(1-x_i)} \leq \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n}(1-x_i)},$$

**Theorem 2.3.** *For all real $x_i \in \left(0, \frac{1}{2}\right]$, $i = 1, 2, \ldots, n$, $(n \geq 2)$, and $1 \leq a \leq \frac{3}{2}$, we have*

$$\frac{\prod_{i=1}^{n}(1+x_i)}{\prod_{i=1}^{n}(1+(1-x_i))} \leq \frac{1 + a\sum_{i=1}^{n} x_i}{1 + a\sum_{i=1}^{n}(1-x_i)}, \tag{2.3}$$

*with equality if and only if $x_i = \frac{1}{2}$ for all $i = 1, 2, \ldots, n$.*

*Proof.* It is easy to see that the equality holds in 2.3 for $x_i = \frac{1}{2}$ where $i = 1, 2, \ldots, n$. Therefore, it remains to prove for numbers $x_1, x_2, \ldots, x_n$ are not all equal to $\frac{1}{2}$ then the Inequality 2.3 is valid with "$<$" instead of "$\leq$".

Let us prove Inequality 2.3 by induction on $n$. For $n=2$ we have to prove, for $0<x_1<x_2\le\frac{1}{2}$,

$$\frac{(1+x_1)(1+x_2)}{(2-x_1)(2-x_2)}<\frac{1+ax_1+ax_2}{1+2a-ax_1-ax_2},$$

equivalently,

$$(3-2a+a(x_1+x_2-2x_1x_2))(x_1+x_2-1),$$

which is absolutely true for $0<x_1<x_2\le\frac{1}{2}$ and $1<a\le\frac{3}{2}$. Now, let us suppose that

$$0<x_1\le x_2\le\cdots\le x_n\le x_{n+1}\le\frac{1}{2},\quad x_1<x_{n+1},$$

and further we suppose that Inequality 2.3 is hold for $n$. Therefore, by induction hypothesis we get

$$\prod_{i=1}^{n+1}\frac{1+x_i}{(1+(1-x_i))}\le\frac{1+a\sum_{i=1}^n x_i}{1+a\sum_{i=1}^n(1-x_i)}\cdot\frac{1+x_{n+1}}{2-x_{n+1}}.$$

If putting $p=a\sum_{i=1}^n x_i$, $q=a\sum_{i=1}^n(1-x_i)$, and $x=x_{n+1}$, then we have to prove that

$$\frac{(1+x)(1+p)}{(2-x)(1+q)}<\frac{1+p+ax}{1+q+a(1-x)},$$

equivalently,

$$a(q-p)x^2+2x(1+q)(1+p-a)+(1+p)(a-1-q)<0.$$

Since, $a=b$ if and only if $x_i=\frac{1}{2}$ for $i=1,2,\ldots,n$, we have $p<q$. Therefore, since $x\le\frac{1}{2}$ and $1<a\le\frac{3}{2}$ we obtain

$$a(q-p)x^2+2x(1+q)(1+p-a)+(1+p)(a-1-q)\le-\frac{3a(q-p)}{4}<0,$$

which this completes the proof. □

For example, Theorem 2.3, for $a=1$ yields the second inequality in [A, Theorem, Equation 2.2].

**Corollary 2.4.** *For all real $x_i\in\left[\frac{1}{2},1\right)$, $i=1,2,\ldots,n$, $(n\ge 2)$, and $1\le a\le\frac{3}{2}$, we have*

$$\frac{\prod_{i=1}^n(1+x_i)}{\prod_{i=1}^n(1+(1-x_i))}\ge\frac{1+a\sum_{i=1}^n x_i}{1+a\sum_{i=1}^n(1-x_i)},\tag{2.4}$$

*with equality if and only if $x_i=\frac{1}{2}$ for all $i=1,2,\ldots,n$.*

*Proof.* If we set $x_i=1-y_i$, $y_i\in\left[\frac{1}{2},1\right)$, $i=1,2,\ldots,n$, in Inequality 2.3 then we arrive to Inequality 2.4. □

**Theorem 2.5.** *For all real $x_i\in\left(0,\frac{1}{2}\right]$, $i=1,2,\ldots,n$, $(n\ge 2)$, and $\frac{2}{3}<a\le 1$, we have*

$$\frac{1+a\sum_{i=1}^n x_i}{1+a\sum_{i=1}^n(1-x_i)}\le\frac{1+\prod_{i=1}^n x_i}{1+\prod_{i=1}^n(1-x_i)},\tag{2.5}$$

*with equality if and only if $x_i=\frac{1}{2}$ for all $i=1,2,\ldots,n$.*

*Proof.* It is easy to see that the equality holds in 2.5 for $x_i = \frac{1}{2}$ where $i = 1, 2, \dots, n$. Therefore, it remains to prove for numbers $x_1, x_2, \dots, x_n$ are not all equal to $\frac{1}{2}$ then the Inequality 2.5 is valid with "$<$" instead of "$\leq$".

Let us prove Inequality 2.5 by induction on $n$. By simplify Inequality 2.5 we arrive to

$$an - 2a\sum_{i=1}^{n} x_i + (an+1)\prod_{i=1}^{n} x_i - \prod_{i=1}^{n}(1-x_i) - a\sum_{i=1}^{n} x_i \left(\prod_{i=1}^{n} x_i + \prod_{i=1}^{n}(1-x_i)\right) \geq 0. \tag{2.6}$$

For $n = 2$ we have to prove, for $0 < x_1 < x_2 \leq \frac{1}{2}$,

$$h(x_2) = 2a - 2a(x_1 + x_2) + (2a+1)x_1x_2 - (1-x_1)(1-x_2) - a(x_1+x_2)(x_1x_2 + (1-x_1)(1-x_2)),$$

where $h : \left(x_1, \frac{1}{2}\right] \to \mathbb{R}$. So, $h''(x_2) = 2x(1-2x_1) > 0$, and $h'(x_2) = 0$ if and only if

$$x_2 = \frac{(x_1-1)^2}{1-2x_1} + \frac{a-1}{2a(1-2x_1)} \geq \frac{(x_1-1)^2}{1-2x_1} - \frac{1}{4(1-2x_1)} = \frac{3}{4} - \frac{1}{2}x_1 \geq \frac{1}{2}.$$

Therefore, $h(x_2)$ is strictly decreasing on $\left(x_1, \frac{1}{2}\right]$ which gives

$$h(x_2) \geq h\left(\frac{1}{2}\right) = \left(\frac{1}{2} - x_1\right)\left(\frac{3a}{2} - 1\right) > 0.$$

Now, let us suppose that

$$0 < x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1} \leq \frac{1}{2}, \quad x_1 < x_{n+1},$$

and further we suppose that Inequality 2.3 is hold for $n$. Let us define a function $f : \left(0, \frac{1}{2}\right] \to \mathbb{R}$ by

$$\begin{aligned} f(x) = a(n+1) - 2a\sum_{i=1}^{n} x_i - 2ax + (an+a+1)x\prod_{i=1}^{n} x_i - (1-x)\prod_{i=1}^{n}(1-x_i) \\ -a\left(x + \sum_{i=1}^{n} x_i\right)\left(x\prod_{i=1}^{n} x_i + (1-x)\prod_{i=1}^{n}(1-x_i)\right), \end{aligned}$$

so we have to prove $f(x_{n+1}) > 0$. This by prove that $f$ is a convex function, $f'\left(\frac{1}{2}\right) < 0$, and $f\left(\frac{1}{2}\right) > 0$. The function $f$ is a convex function because $f''(x) = 2a\left(\prod_{i=1}^{n}(1-x_i) - \prod_{i=1}^{n} x_i\right) \geq 0$. By direct calculations together with Lemma 2.1 we arrive at

$$f'\left(\frac{1}{2}\right) = \prod_{i=1}^{n} x_i\left(1 + a\sum_{i=1}^{n} x_i\right) + \prod_{i=1}^{n}(1-x_i)\left(1 + a\sum_{i=1}^{n} x_i\right) - 2a \leq 1 + \frac{a}{2} - 2a \leq 1 - \frac{3a}{2} < 0.$$

To prove that $f\left(\frac{1}{2}\right) > 0$, let us write

$$f\left(\frac{1}{2}\right) = u(x_1, \dots, x_n) + v(x_1, \dots, x_n) + \frac{1}{2}(1-a)\left(\prod_{i=1}^{n}(1-x_i) - \prod_{i=1}^{n} x_i\right),$$

where

$$\begin{aligned} &u(x_1, \dots, x_n) \\ &\quad = an - 2a\sum_{i=1}^{n} + (an+1)\prod_{i=1}^{n} x_i - \prod_{i=1}^{n}(1-x_i) - a\sum_{i=1}^{n} x_i\left(\prod_{i=1}^{n} x_i + \prod_{i=1}^{n}(1-x_i)\right), \\ &v(x_1, \dots, x_n) \\ &\quad = \tfrac{a}{4}\prod_{i=1}^{n}(1-x_i)\left(1 + 2\sum_{i=1}^{n} x_i\right) - \tfrac{a}{4}\prod_{i=1}^{n} x_i\left(1 + 2\sum_{i=1}^{n}(1-x_i)\right). \end{aligned}$$

Since $\frac{2}{3} < a \leq 1$ we have $\frac{1}{2}(1-a)\left(\prod_{i=1}^{n}(1-x_i) - \prod_{i=1}^{n} x_i\right) \geq 0$, and by induction hypothesis we get $u(x_1,\ldots,x_n) \geq 0$. Lemma 2.2, for $a = 2$, yields $v(x_1,\ldots,x_n) > 0$. Hence, $f\left(\frac{1}{2}\right) > 0$ as requested. □

For example, Theorem 2.5, for $a = 1$ yields the third inequality in [A, Theorem, Equation 2.2].

**Corollary 2.6.** *For all real* $x_i \in \left[\frac{1}{2}, 1\right)$, $i = 1, 2, \ldots, n$, $(n \geq 2)$, *and* $1 \leq a \leq \frac{3}{2}$, *we have*

$$\frac{1 + a\sum_{i=1}^{n} x_i}{1 + a\sum_{i=1}^{n}(1-x_i)} \geq \frac{1 + \prod_{i=1}^{n} x_i}{1 + \prod_{i=1}^{n}(1-x_i)}, \tag{2.7}$$

*with equality if and only if* $x_i = \frac{1}{2}$ *for all* $i = 1, 2, \ldots, n$.

*Proof.* If we set $x_i = 1 - y_i$, $y_i \in \left[\frac{1}{2}, 1\right)$, $i = 1, 2, \ldots, n$, in Inequality 2.5 then we arrive to Inequality 2.7. □

## 3. Applications

In this section we present two applications of the pervious results. In [M] presented a great number of inequalities involving trigonometric functions. Here, we can be formulated another two inequalities which are the analogue of [A, Corollary 1].

**Corollary 3.1.**

(i) *For all* $x_i \in \left(0, \frac{\pi}{4}\right]$, $i = 1, 2, \ldots, n$, $(n \geq 2)$, *and* $1 \leq a \leq \frac{3}{2}$, *we have*

$$\frac{\prod_{i=1}^{n}(1 + \sin^2(x_i))}{\prod_{i=1}^{n}(1 + \cos^2(x_i))} \leq \frac{1 + a\sum_{i=1}^{n}\sin^2(x_i)}{1 + a\sum_{i=1}^{n}\cos^2(x_i)},$$

(ii) *For all* $x_i \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right)$, $i = 1, 2, \ldots, n$, $(n \geq 2)$, *and* $1 \leq a \leq \frac{3}{2}$, *we have*

$$\frac{\prod_{i=1}^{n}(1 + \sin^2(x_i))}{\prod_{i=1}^{n}(1 + \cos^2(x_i))} \geq \frac{1 + a\sum_{i=1}^{n}\sin^2(x_i)}{1 + a\sum_{i=1}^{n}\cos^2(x_i)},$$

*Proof.* If we replace $x_i$ by $\sin^2(x_i)$ for all $i = 1, 2, \ldots, n$, then the corollary follow immediately from Theorem 2.2 and Corollary 2.4. □

For example, Corollary 3.1, for $a = 1$, yields [A, Corollary 1].

An another application for Theorem 2.3 and Corollary 2.4 we can present the following result.

**Corollary 3.2.**

(i) *For any integrable function* $f : [0, n] \to \left(0, \frac{1}{2}\right]$ *where* $n \geq 2$, *and* $1 \leq a \leq \frac{3}{2}$, *we have*

$$\frac{\prod_{i=1}^{n}\int_{i-1}^{i}(1 + f(t))dt}{\prod_{i=1}^{n}\int_{i-1}^{i}(2 - f(t))dt} \leq \frac{1 + a\int_0^n f(t)dt}{1 + a\int_0^n(1 - f(t))dt},$$

(ii) *For any integrable function* $f : [0, n] \to \left[\frac{1}{2}, 1\right)$ *where* $n \geq 2$, *and* $1 \leq a \leq \frac{3}{2}$, *we have*

$$\frac{\prod_{i=1}^{n}\int_{i-1}^{i}(1 + f(t))dt}{\prod_{i=1}^{n}\int_{i-1}^{i}(2 - f(t))dt} \geq \frac{1 + a\int_0^n f(t)dt}{1 + a\int_0^n(1 - f(t))dt},$$

*Proof.* If setting $x_i = \int_{i-1}^{i} f(t)dt$, then the corollary immediately follow from the Theorem 2.3 and Corollary 2.4. □

For example, Corollary 3.2, for $a = 1$, yields [A, Corollary 2].

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