Pattern avoidance

via functional equations

or

Generating trees with two labels

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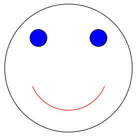
Enumeration: what is a hard problem?



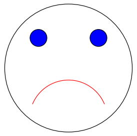
Recursive description of the objects

$\longrightarrow$ functional equation that defines the generating function

$\longrightarrow$ explicit expression for the GF, or for the numbers, asymptotic behaviour...

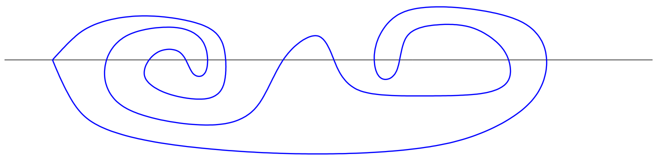
Examples: Dyck paths, labelled graphs...

Simple decomposable structures



No recursive description, no obvious structure.

Examples: self-avoiding walks, meanders...



Enumeration with catalytic parameters

• There exists a recursive structure, but, in order to exploit it (i.e. write a functional equation), one needs to take into account a (finite) number of additional parameters.

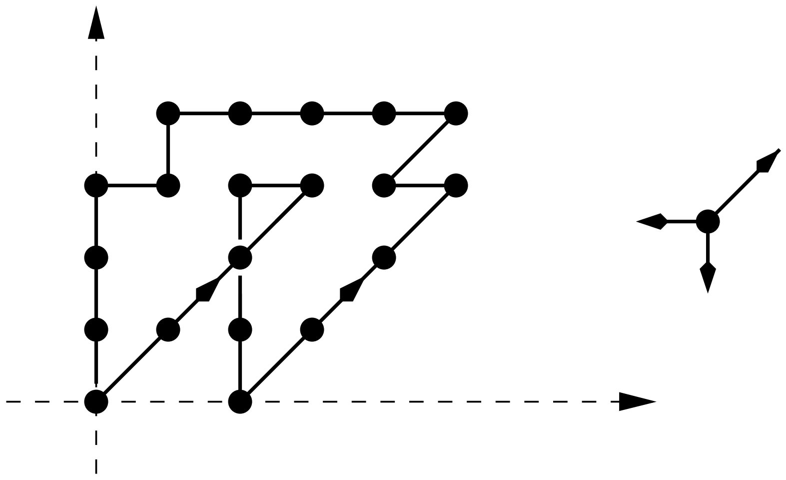
$\Rightarrow$ generating function in several variables $F(t; u, v, \ldots)$.

The variables u and v are said to be catalytic [Zeilberger].

• Combinatorics is done, but gives a strange equation.

What can be said about $F(t; 1, 1, \ldots)$? What about its coefficients?

Example: Kreweras' walks in the quadrant



The number of walks of length $3n$ starting and ending at the origin is

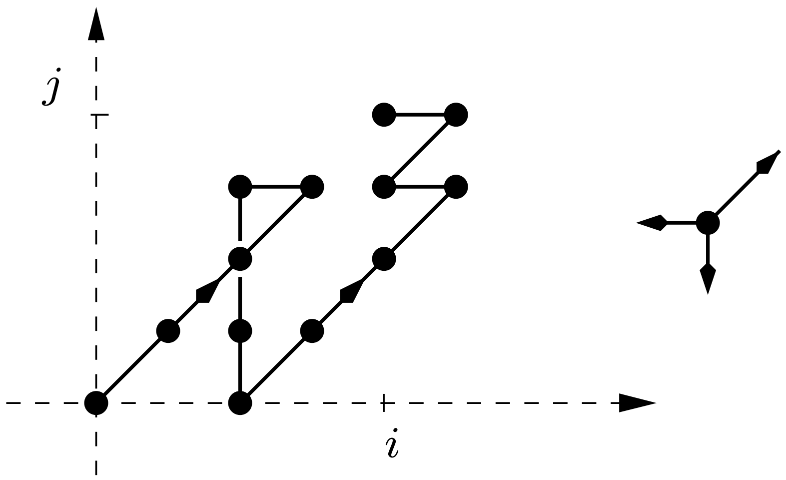
$$a(3n) = \frac{4^n}{(n+1)(2n+1)}\binom{3n}{n}.$$

[Kreweras 65]

The generating function $A(t) = \sum a(3n)t^{3n}$ is algebraic:

$$\text{Let} \qquad W = t(2+W^3) \quad \text{then} \qquad A = \frac{W}{2t}\left(1 - \frac{W^3}{4}\right).$$

Kreweras' walks via catalytic variables



Three-variable generating function: $Q(t;u,v) \equiv Q(u,v) = \sum_{n,i,j} a_{i,j}(n) t^n u^i v^j$

Step-by-step construction:

$$Q(u,v) = 1 + tuvQ(u,v) + t\,\frac{Q(u,v) - Q(0,v)}{u} + t\,\frac{Q(u,v) - Q(u,0)}{v}$$

that is,

$$\left(uv - t(u + v + u^2v^2)\right) Q(u,v) = uv - tuQ(u,0) - tvQ(0,v).$$

How can we compute $Q(0,0)$? Why is it algebraic? What about $Q(u,v)$?

Algebraic series, D-finite series

• The series $F(t,u,v)$ is algebraic if there exists a non-trivial polynomial P such that

$$P(F,t,u,v)=0.$$

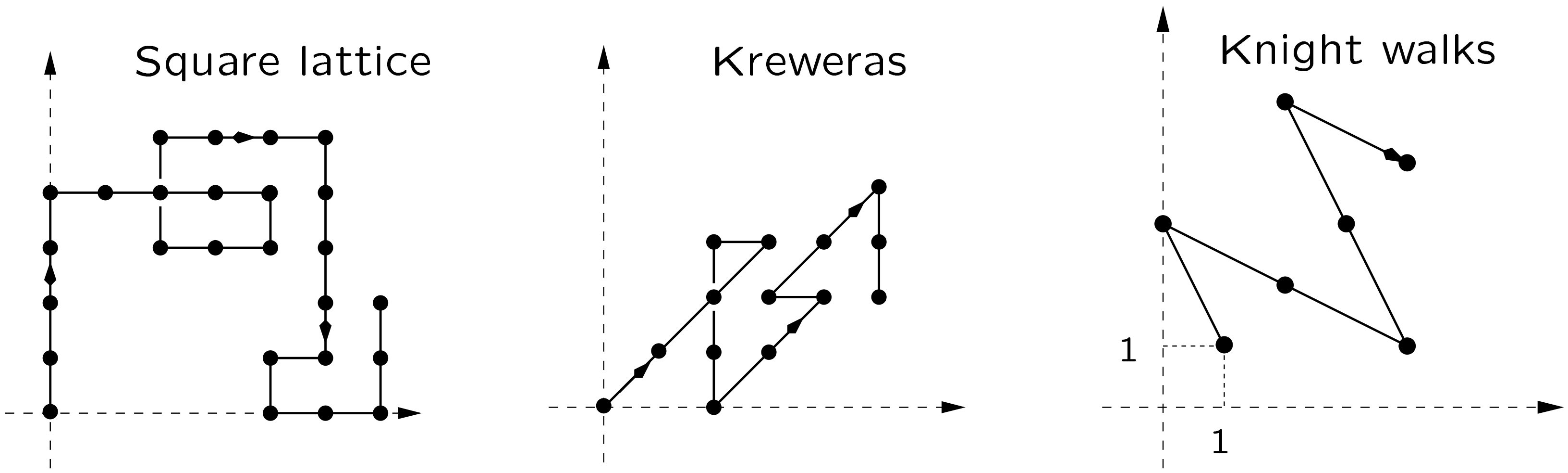
• It is D-finite (or : holonomic) if it satisfies three linear differential equations with polynomial coefficients (one per variable):

$$P_0(t,u,v)F(t,u,v)+P_1(t,u,v)\frac{\partial F}{\partial t}(t,u,v)+\cdots+P_k(t,u,v)\frac{\partial^k F}{\partial t^k}(t,u,v)=0,$$

and similarly for the variables u and v.

• Every algebraic series is D-finite.

Variations on the theme



1. D-finite and transcendental

$$(uv - t(u+v)(1+uv))\, Q(u,v) = uv - tuQ(u,0) - tvQ(0,v)$$

2. Algebraic

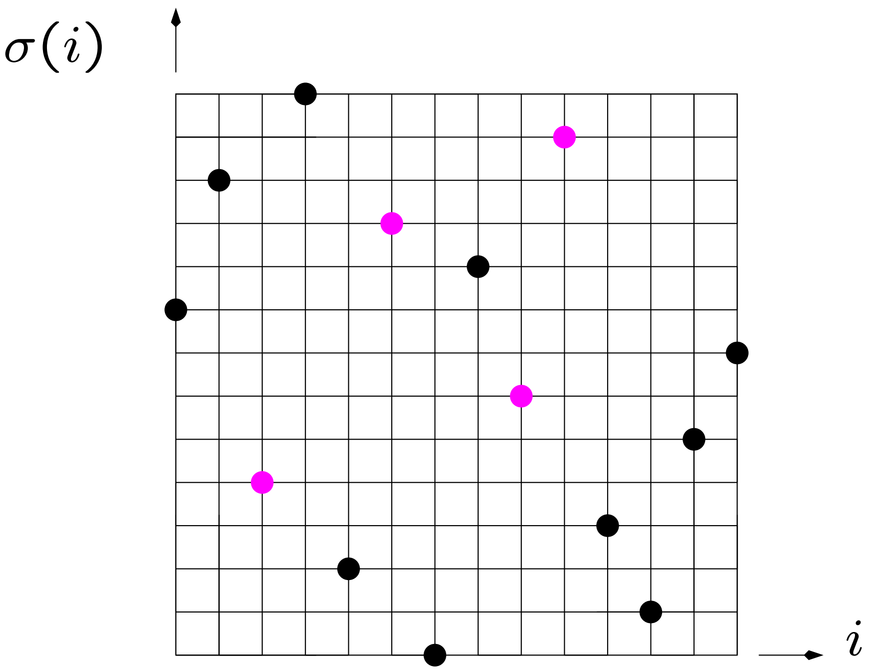
$$\left(uv - t(u+v+u^2v^2)\right) Q(u,v) = uv - tuQ(u,0) - tvQ(0,v)$$

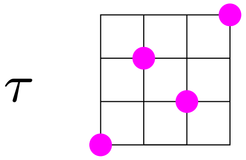
3. Not D-finite (infinitely many singularities)

$$\left(uv - t(u^3+v^3)\right) Q(u,v) = u^2v^2 - tu^3Q(u,0) - tv^3Q(0,v)$$

[MBM 02, MBM-Petkovšek 03]

Pattern avoidance in permutations





The permutation $\sigma = 9.12.5.14.3.11.1.10.7.13.4.2.6.8$ contains the pattern $\tau = 1324$.

Let $a_n(\tau)$ denote the number of permutations of $\mathcal{S}_n$ that avoid the pattern τ.

In some cases...

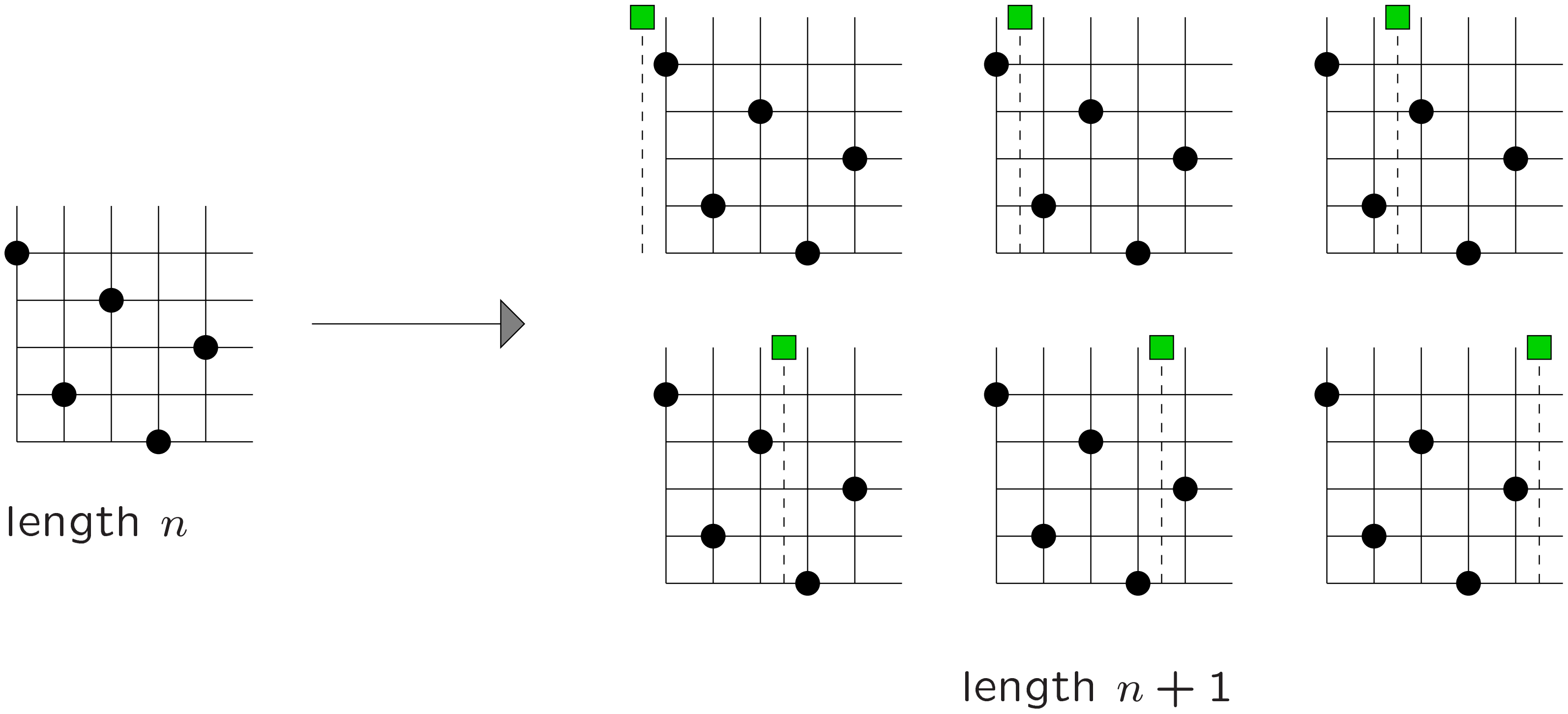
• Simple recursive description... that requires to add a few parameters $\Rightarrow$ functional equation with catalytic variables.

Example: involutions avoiding 2143 (vexillary)
1995: recursive structure with two catalytic parameters, and simple conjecture (Motzkin)
2001: proof ($\sim$ bijective) [Guibert-Pergola-Pinzani]

$$M_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1}{k+1} \binom{2k}{k} \binom{n}{k}$$

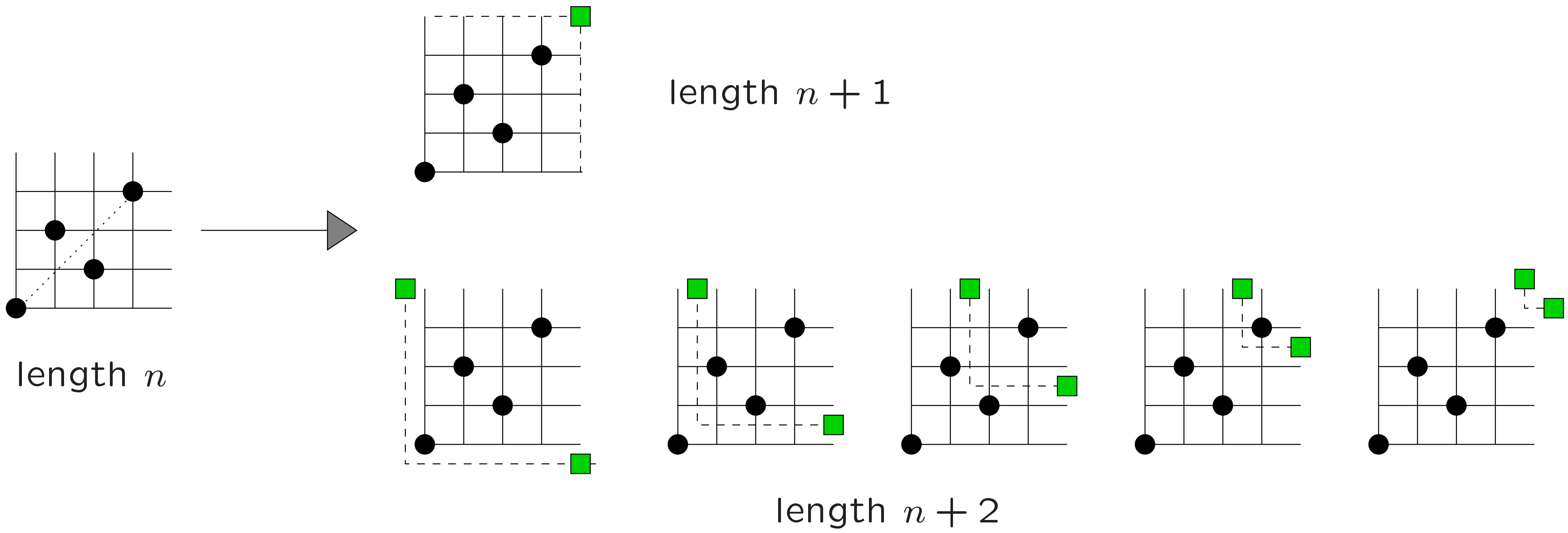
$$\sum_{n\geqslant 0} M_n t^n = \frac{1 - t - \sqrt{(1+t)(1-3t)}}{2t^2}$$

A recursive construction for permutations



Insertion of $(n+1)$ in a permutation of length n.

A recursive construction for involutions



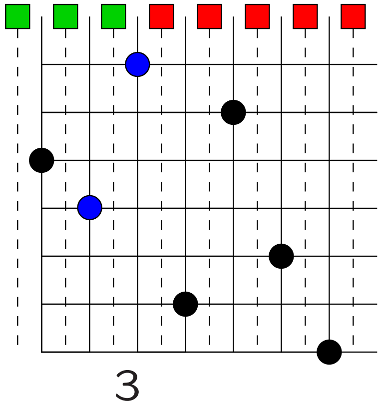
Insertion of a fixed point, or of a 2-cycle.

Ex. 0: Permutations avoiding 123

allowed insertion

forbidden insertion

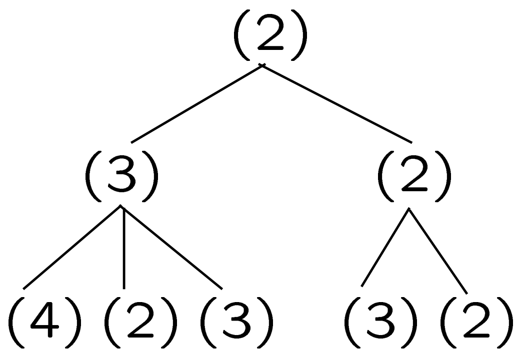
$p = 3$



The new letter n must be inserted before the first rise

Let p denote the position of the first rise:

$$(p) \to (p+1), (2), (3), \ldots, (p)$$



Functional equation for 123-avoiding permutations

Generating function:

$$G(t;u) \equiv G(u) = \sum_{\sigma\in\mathcal{S}(123)} t^{\ell(\sigma)} u^{p(\sigma)} = \sum_{p\geqslant 1} G_p(t) u^p$$

$$(p) \to (p+1), (2), (3), \ldots, (p)$$

$$\begin{aligned} G(u) &= u + t\sum_{p\geqslant 1} G_p(t)(u^2 + \cdots + u^p + u^{p+1}) \\ &= u + tu^2\,\frac{G(u)-G(1)}{u-1}, \end{aligned}$$

that is,

$$\left(1+\frac{tu^2}{1-u}\right)G(u) = u + \frac{tu^2}{1-u}G(1).$$

The kernel method

$$\left(1+\frac{tu^2}{1-u}\right)G(u) = u + \frac{tu^2}{1-u}G(1).$$

- The kernel $K(u) = 1 - u + tu^2$ has two roots:

$$U_{0,1}(x) = \frac{1 \mp \sqrt{1-4t}}{2t}.$$

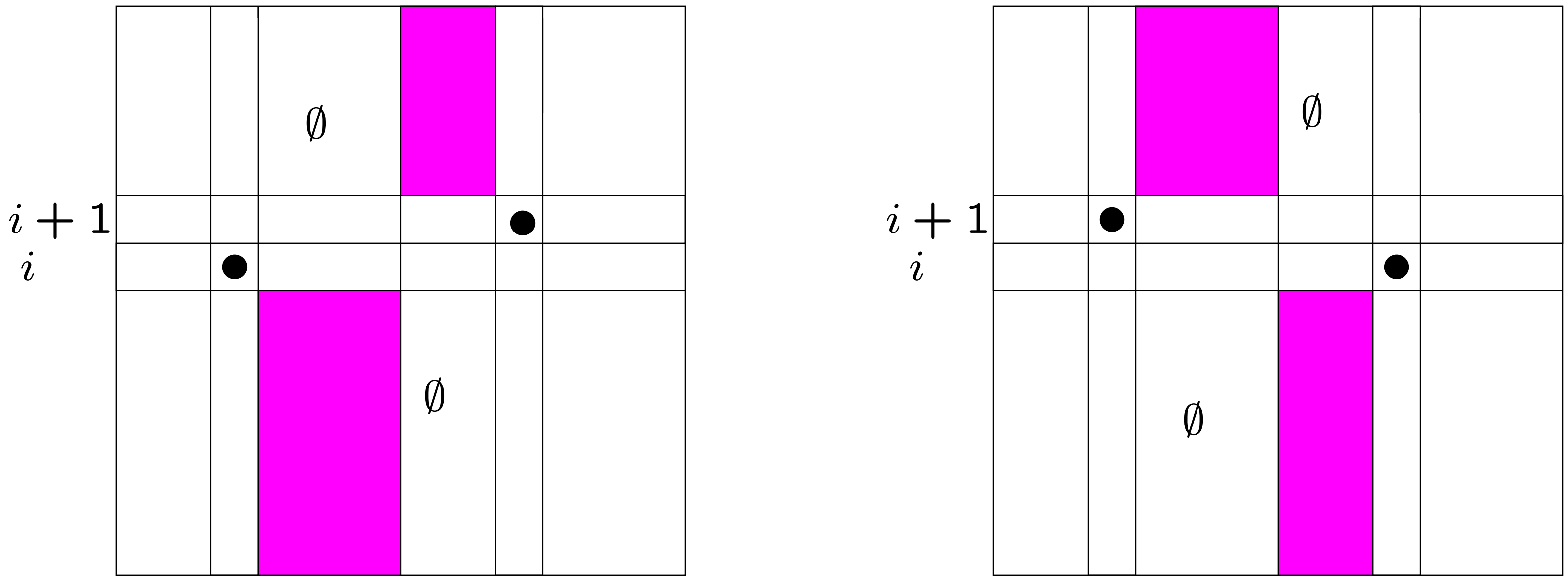
Then

$$U_0 = 1 + O(t) \quad \text{and} \quad U_1 = \frac{1}{t} + O(1).$$

- One can substitute U_0 for u in the equation (but not U_1) :

$$G(1) = U_0 \quad \text{et} \quad a_n(123) = C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Ex. 1: the Baxter permutations



$\sigma^{-1}(i) < \sigma^{-1}(i+1)$ $\qquad\qquad$ $\sigma^{-1}(i+1) < \sigma^{-1}(i)$

For all $i \in \{1, \ldots, n-1\}$, the word σ can be written

$$\sigma = \pi \ i \ \pi_- \ \pi_+ \ (i+1) \ \pi' \quad \text{or} \quad \sigma = \pi \ (i+1) \ \pi_+ \ \pi_- \ i \ \pi',$$

where all the letters occurring in π_+ (resp. π_-) are larger (resp. smaller) than i.

The number of Baxter permutations

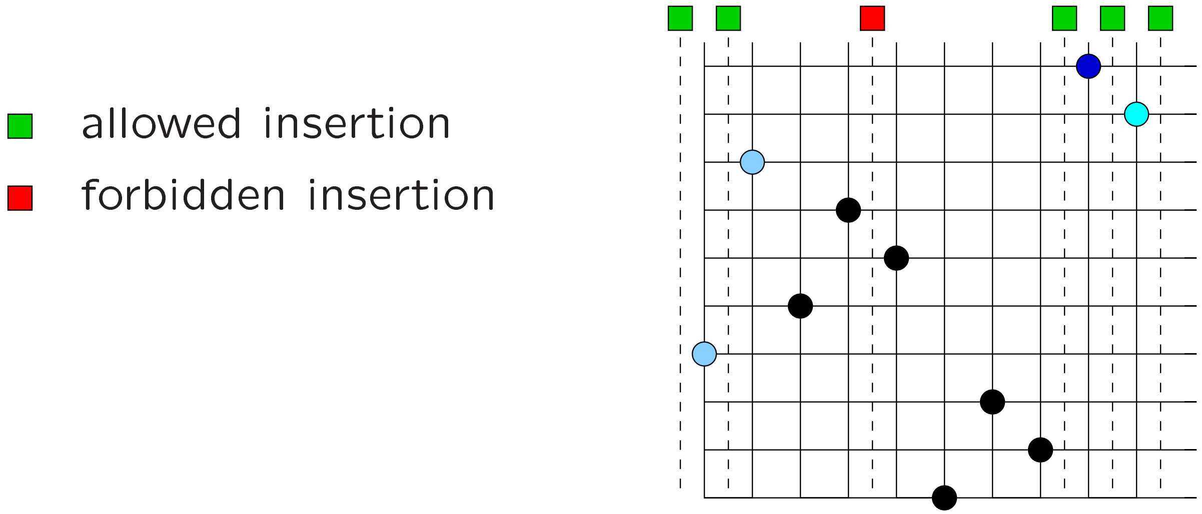
The number of Baxter permutations of length n is:

$$b(n) = \frac{2}{n(n+1)^2} \sum_{k=1}^{n} \binom{n+1}{k-1}\binom{n+1}{k}\binom{n+1}{k+1}.$$

The series $\sum_n b(n)t^n$ is D-finite.

[Chung-Graham-Hoggatt-Kleiman 78, Mallows 79, Viennot 81, Cori-Dulucq-Viennot 86, Dulucq-Guibert 96]

Recursive construction of Baxter permutations



Insertion just before a left-to-right maximum (*lrm*) or just after a right-to-left maximum (*rlm*).

If σ contains p *lrm* and q *rlm*, one obtains $p+q$ Baxter permutations:

$$(p,q) \to \begin{cases} (1,q+1),(2,q+1),\dots,(p,q+1) \\ (p+1,q),(p+1,q-1),\dots,(p+1,1). \end{cases}$$

Functional equation for Baxter permutations

Generating function: $$G(t;u,v) \equiv G(u,v) = \sum_{\sigma\in\mathcal{B}} t^{\ell(\sigma)} u^{p(\sigma)} v^{q(\sigma)}$$

$$(p,q) \rightarrow \begin{cases} (1,q+1),(2,q+1),\ldots,(p,q+1) \\ (p+1,q),(p+1,q-1),\ldots,(p+1,1). \end{cases}$$

$$\begin{aligned} G(u,v) &= tuv + t\sum_{p,q\geqslant 1} G_{p,q}(t)\Big((u+u^2+\cdots+u^p)v^{q+1} \\ &\qquad\qquad + u^{p+1}(v^q+v^{q-1}+\cdots+v)\Big) \\ &= tuv + tuv\frac{G(u,v)-G(1,v)}{u-1} + tuv\frac{G(u,v)-G(u,1)}{v-1} \end{aligned}$$

$$\left(1+\frac{tuv}{1-u}+\frac{tuv}{1-v}\right)G(u,v) = tuv + \frac{tuv}{1-v}G(u,1) + \frac{tuv}{1-u}G(1,v).$$

Step 1: the kernel method

- Let $u = 1 + x$, $v = 1 + y$.

$$\frac{(1+x)(1+y)(x+y)t - xy}{t(1+x)(1+y)} G(1+x, 1+y) = R(x) + R(y) - xy$$

with $R(x) = xG(1+x, 1)$.

- The kernel $K(x,y) = (1+x)(1+y)(x+y)t - xy$ has two roots:

$$Y_{0,1}(x) = \frac{1 - t(1+x)(1+\bar{x}) \mp \sqrt{1 - 2t(1+x)(1+\bar{x}) - t^2(1-x^2)(1-\bar{x}^2)}}{2t(1+\bar{x})}$$

with $\bar{x} = 1/x$. Then

$$Y_0 = (1+x)t + O(t^2) \qquad \text{and} \qquad Y_1 = \frac{x}{1+x}\frac{1}{t} + O(1).$$

- One can substitute Y_0 for y in the equation (but not Y_1) :

$$R(x) + R(Y_0) = xY_0.$$

- This equation defines $R(x)$ uniquely among the series in t with coefficients dans $x\mathbb{Q}[x]$.

Step 2: let's be obstinate!

Let (X, Y) be a pair of series that cancels the kernel. Define $\Phi(X, Y) = (X', Y)$, where X' is the other root of $K(x, Y) = 0$. Similarly, define $\Psi(X, Y) = (X, Y')$, where Y' is the other root of $K(X, y) = 0$. The maps Φ and Ψ are involutions, and the orbit of (x, Y_0) under their action is finite:

$$K(x, y) = (1 + x)(1 + y)(x + y)t - xy$$

$$\bar{x} := 1/x$$

$$Y_0 Y_1 = x$$

$$\begin{tikzcd}
 & \boxed{(x, Y_0)} \arrow[dl, "\Phi"'] \arrow[dr, "\Psi"] & \\
\boxed{(\bar{x}Y_0, Y_0)} \arrow[d, "\Psi"'] & & (x, Y_1) \arrow[d, "\Phi"] \\
\boxed{(\bar{x}Y_0, \bar{x})} \arrow[dr, "\Phi"'] & & (\bar{x}Y_1, Y_1) \arrow[dl, "\Psi"] \\
 & (\bar{x}Y_1, \bar{x}) &
\end{tikzcd}$$

[Fayolle-Iasnogorodski-Malyshev 91]

Step 3: playing with the equations

We now have three equations:

$$\left\{\begin{array}{lcl} R(x)+R(Y_0) & = & xY_0, \\ R(\bar{x}Y_0)+R(Y_0) & = & \bar{x}Y_0^2 \\ R(\bar{x}Y_0)+R(\bar{x}) & = & \bar{x}^2Y_0. \end{array}\right.$$

Linear combination:

$$R(x)+R(\bar{x})=\bar{x}^2Y_0(1+x^3-xY_0).$$

The series $R(x)=xG(1+x,1)$ only contains positive powers of x, while $R(\bar{x})$ only contains negative powers of x. Hence

$$R(x)=\left(\bar{x}^2Y_0(1+x^3-xY_0)\right)^{>}$$

The series $R(x)=xG(1+x,1)$ is the positive part (in x) of an algebraic series (in t, with coefficients in $\mathbb{Q}[x,\bar{x}]$): it is thus D-finite [Lipshitz 88].

Step 4: the Lagrange inversion formula

The series Y_0 is defined by

$$Y_0 = t(1+\bar{x})(1+Y_0)(x+Y_0).$$

One has

$$R(x) = xG(1+x,1) = \left(\bar{x}^2 Y_0(1+x^3 - xY_0)\right)^{>}.$$

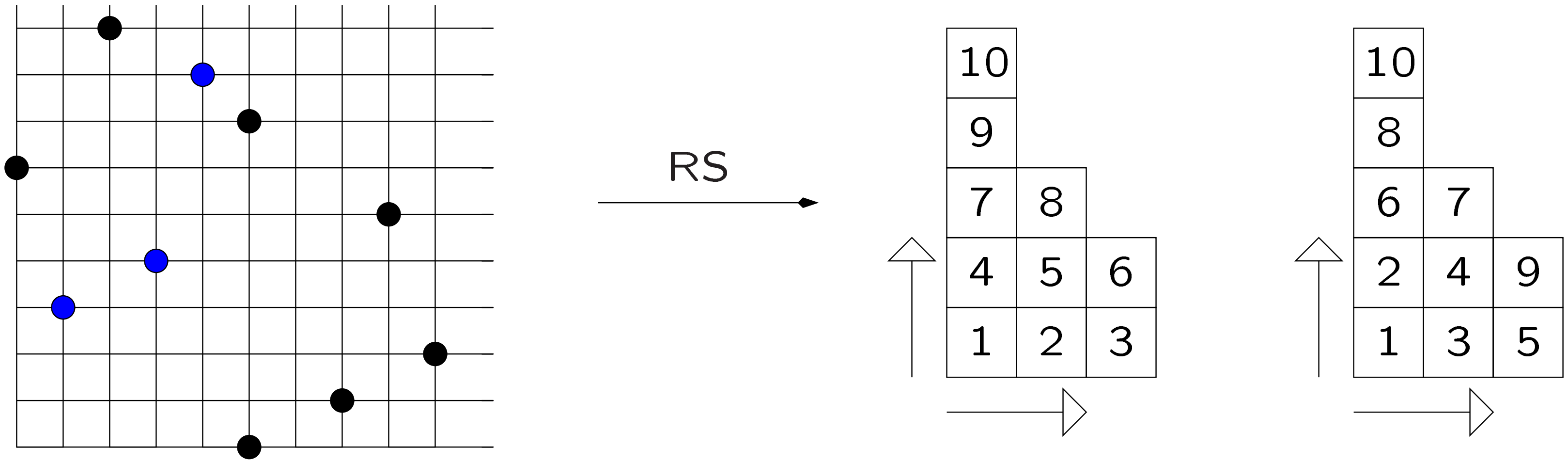
In particular,

$$xG(1,1) = [x]\left(\bar{x}^2 Y_0(1+x^3 - xY_0)\right)$$

Extracting the coefficient of x gives:

$$G(1,1) = \sum_{n\geqslant 1} t^n \frac{2}{n(n+1)^2} \sum_{k=1}^{n} \binom{n+1}{k-1}\binom{n+1}{k}\binom{n+1}{k+1}.$$

Ex. 2: 1234-avoiding permutations



Robinson-Schensted: $a_n(1234)$ is the number of pairs of standard Young tableaux having the same shape and width at most 3.
Symmetric functions $\rightarrow$ determinantal formula [Gessel 90, Stanley 99]
Combinatorial interpretation (paths in $\mathbb{Z}^d$) [Gessel-Weinstein-Wilf 98]

The formula:

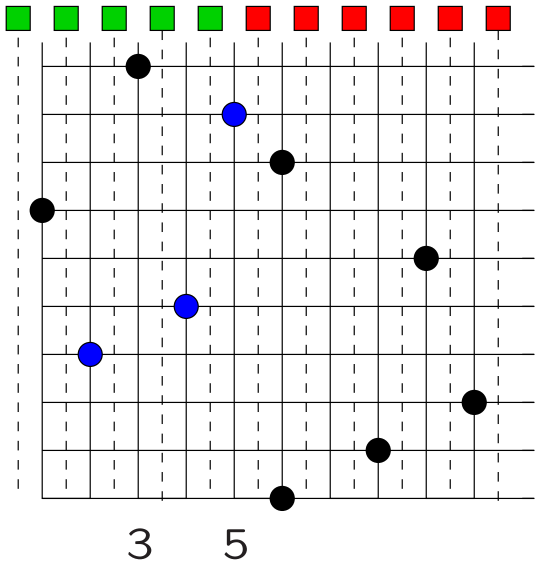
$$a_n(1234) = \frac{1}{(n+1)^2(n+2)} \sum_{k=0}^{n} \binom{2k}{k} \binom{n+1}{k+1} \binom{n+2}{k+1}.$$

Construction of 1234-avoiding permutations

■ allowed insertion

■ forbidden insertion

$q = 5,\ p = 3$



The new letter n must be inserted before the end of the first increasing sub-sequence of length 3 (position q).

Let p denote the position of the first rise:

$$(p, q) \rightarrow \begin{cases} (p+1, q+1), (2, q+1), \ldots, (p, q+1) \\ (p, p+1), (p, p+2), \ldots, (p, q). \end{cases}$$

Functional equation for 1234-avoiding permutations

Generating function:

$$G(t;u,v) \equiv G(u,v) = \sum_{\sigma \in \mathcal{S}(1234)} t^{\ell(\sigma)} u^{p(\sigma)} v^{q(\sigma)}$$

$$(p,q) \to \begin{cases} (p+1,q+1),(2,q+1),\ldots,(p,q+1) \\ (p,p+1),(p,p+2),\ldots,(p,q). \end{cases}$$

$$\left(1+\frac{tu^2v}{1-u}+\frac{tv}{1-v}\right)G(u,v) = uv + \frac{tv}{1-v}G(uv,1) + \frac{tu^2v}{1-u}G(1,v).$$

Remark. Same equation for the enumeration of 1243-avoiding permutations and 2143-avoiding permutations [West 95]. Bijection with 4123-avoiding permutations [Stankova 96].

A change of variables

$$\left(1+\frac{tu^2v}{1-u}+\frac{tv}{1-v}\right)G(u,v)=uv+\frac{tv}{1-v}G(uv,1)+\frac{tu^2v}{1-u}G(1,v).$$

Let $uv=w$

$$\left(1+\frac{tw^2}{v-w}+\frac{tv}{1-v}\right)G(w/v,v)=w+\frac{tv}{1-v}G(w,1)+\frac{tw^2}{v-w}G(1,v),$$

then $w=1+\bar{x}$ and $v=1+y$:

$$\left(xy(1-xy)-t(x+y+3xy-x^2y^2)\right)G\left(\frac{1+\bar{x}}{1+y},1+y\right)$$

$$=y(1+x)(1-xy)-t(1+y)(1-xy)R(\bar{x})-t(x+1)^2S(y)$$

with $R(\bar{x})=xG(1+\bar{x},1)$ and $S(y)=yG(1,1+y)$.

The kernel and its roots

The kernel $K(x,y) = xy(1-xy) - t(x+y+3xy-x^2y^2)$ is symmetric in x et y, and has two roots:

$$\begin{aligned}
Y_0(x) &= \frac{1 - t\bar{x} - 3t - \sqrt{(1-t-t\bar{x})(1-5t-4tx-t\bar{x})}}{2x(1-t)}, \\
&= t + (\bar{x}+3+x)t^2 + O(t^3),
\end{aligned}$$

$$\begin{aligned}
Y_1(x) &= \frac{1 - t\bar{x} - 3t + \sqrt{(1-t-t\bar{x})(1-5t-4tx-t\bar{x})}}{2x(1-t)}, \\
&= \bar{x} - (1+\bar{x})^2 t + O(t^2).
\end{aligned}$$

The elementary symmetric functions of Y_0 and Y_1 are polynomials in $\bar{x} = 1/x$:

$$Y_0 + Y_1 = \frac{\bar{x}(1 - t\bar{x} - 3t)}{1-t} \quad \text{and} \quad Y_0 Y_1 = \frac{t\bar{x}}{1-t}.$$

The diagram of the roots

$$
\begin{tikzcd}
& \boxed{(x, Y_0)} \arrow[dl, "\Phi"'] \arrow[dr, "\Psi"] & \\
\boxed{(Y_1, Y_0)} \arrow[d, "\Psi"'] & & \boxed{(x, Y_1)} \arrow[d, "\Phi"] \\
\boxed{(Y_1, x)} \arrow[dr, "\Phi"'] & & (Y_0, Y_1) \arrow[dl, "\Psi"] \\
& (Y_0, x) &
\end{tikzcd}
$$

$$K(x, y) = K(y, x)$$

$$xY_0Y_1 = \frac{t}{1-t}$$

The four equations

$$\begin{cases} t(1+Y_0)(1-xY_0)R(\bar{x}) & + & t(1+x)^2S(Y_0) & = & Y_0(1+x)(1-xY_0), \\ t(1+Y_1)(1-xY_1)R(\bar{x}) & + & t(1+x)^2S(Y_1) & = & Y_1(1+x)(1-xY_1), \\ t(1+Y_0)(1-Y_1Y_0)R(\bar{Y}_1) & + & t(Y_1+1)^2S(Y_0) & = & Y_0(1+Y_1)(1-Y_1Y_0), \\ t(1+x)(1-Y_1x)R(\bar{Y}_1) & + & t(Y_1+1)^2S(x) & = & x(1+Y_1)(1-Y_1x). \end{cases}$$

Elimination of $R(\bar{x})$ and $R(\bar{Y}_1)$ $\rightarrow$ two equations relating $S(x), S(Y_0)$ and $S(Y_1)$.

The symmetric functions of Y_0 and Y_1 only contain negative powers of x:

$$Y_0 + Y_1 = \frac{\bar{x}(1 - t\bar{x} - 3t)}{1 - t} \quad \text{and} \quad Y_0 Y_1 = \frac{t\bar{x}}{1 - t}.$$

$\Rightarrow$ Form linear combinations of the two equations so as to obtain equations in which Y_0 and Y_1 play symmetric roles: a divided difference,

$$\frac{(1 + Y_1)(1 - xY_1)S(Y_0) - (1 + Y_0)(1 - xY_0)S(Y_1)}{Y_0 - Y_1} = \frac{1 + \bar{x}}{1 - t}$$

and a good old sum,

$$t(x+1)((1+Y_1)(1-xY_1)S(Y_0)+(1+Y_0)(1-xY_0)S(Y_1)) =$$

$$2t(1+Y_0)(1+Y_1)(1-Y_0Y_1)S(x)-(1-xY_1)(2x-Y_0-Y_1-Y_0Y_1x-xY_0^2+2Y_0^2Y_1).$$

$\Rightarrow$ Extract the positive part in x.

The result

Let $A(t;x)$ be the following formal power series in t:

$$A(t;x) \equiv A(x) = \frac{\bar{x}^2(1-2t\bar{x}-t-x+xt)}{2(1-t-t\bar{x})}\sqrt{\frac{1-t\bar{x}-5t-4xt}{1-t-t\bar{x}}}.$$

Its coefficients belong to $\mathbb{Q}[x,\bar{x}]$, with $\bar{x}=1/x$.

The series $G(t;1,v)$ which counts 1234-avoiding permutations by the length and position of the first 123 pattern satisfies

$$G(t;1,1+x) = 1 + [x^0]A(x) + \frac{x}{t}A^{\geqslant}(x).$$

In particular, $G(t;1,1) = 1 + [x^0]A(x)$.

Lagrange inversion formula:

$$a_n(1234) = \frac{1}{(n+1)^2(n+2)}\sum_{k=0}^{n}\binom{2k}{k}\binom{n+1}{k+1}\binom{n+2}{k+1}.$$

Ex. 3: Vexillary involutions (2143-avoiding)

- Functional equation

$$\left(1+\frac{t^2u^2v}{1-u}+\frac{t^2v}{1-v}\right)G(u,v)=\frac{uv(1-t)}{1-uvt}+t\left(1+\frac{tv}{1-v}\right)G(uv,1)+\frac{t^2u^2v}{1-u}G(1,v).$$

cf.1234-avoiding permutations

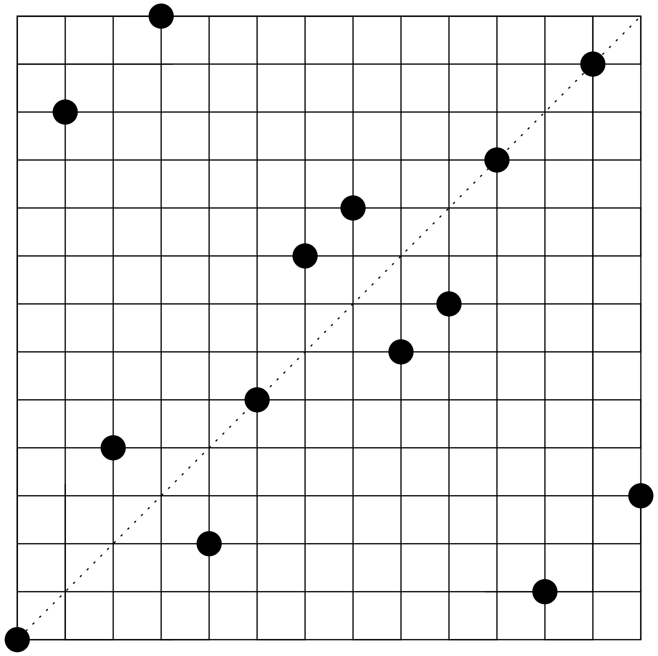
$$\left(1+\frac{tu^2v}{1-u}+\frac{tv}{1-v}\right)G(u,v)=uv+\frac{tv}{1-v}G(uv,1)+\frac{tu^2v}{1-u}G(1,v).$$

- Very simple solution: $G(u,v)$ is algebraic of degree 2, and

$$G(1,1)=\sum_{n\geqslant 0}t^n\sum_{k=0}^{\lfloor n/2\rfloor}\frac{1}{k+1}\binom{2k}{k}\binom{n}{k}.$$

Motzkin numbers [Guibert-Pergola-Pinzani 01]

Ex. 4: 54321-avoiding involutions



$\sigma = 1.12.5.14.3.6.9.10.7.8.11.2.13.4$

$\xrightarrow{\text{RS}}$

12	14					
5	9					
3	6	10				
1	2	4	7	8	11	13

Robinson-Schensted: standard Young tableaux of height at most 4.
Combinatorial solution [Gouyou-Beauchamps 89]
Symmetric functions [Gessel 90, Stanley 99]

- Functional equation

$$\left(1+\frac{t^2u^2v^2}{1-u}+\frac{t^2uv^2}{1-v}\right)G(u,v)=uv+\frac{t^2uv^2}{1-v}G(uv,1)+tuv\left(1+\frac{tuv}{1-u}\right)G(1,v).$$

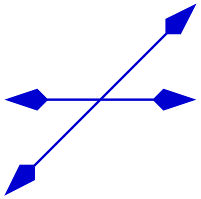
Obstinate kernel method + extraction of a positive part:

$$G(1,1) = \sum_{n \geqslant 0} t^n C_{\lceil n/2 \rceil} C_{1+\lfloor n/2 \rfloor}$$

where C_n is the nth Catalan number.

Generation function of 54321-avoiding involutions.

Other nice equations (1)



$(1,0), (1,1)$

$(-1,0), (-1,-1)$

- Walks in the quarter plane

$$(uv - t(1+u)(1+v))Q(u,v) = uv - tQ(u,0) - t(1+v)Q(0,v) + tQ(0,0)$$

- Number of walks returning to the origin:

$$\begin{aligned} a_{2n}(0,0) &= 4^n \frac{1.3.5\dots(2n-1)}{(n+1)!} \frac{5.11\dots(6n-1)}{5.8\dots(3n-1)(3n+2)} \\ &= 4^{2n} \frac{\Gamma(5/3)}{\Gamma(1/2)\Gamma(5/6)} \frac{\Gamma(n+1/2)\Gamma(n+5/6)}{\Gamma(n+5/3)\Gamma(n+2)} \end{aligned}$$

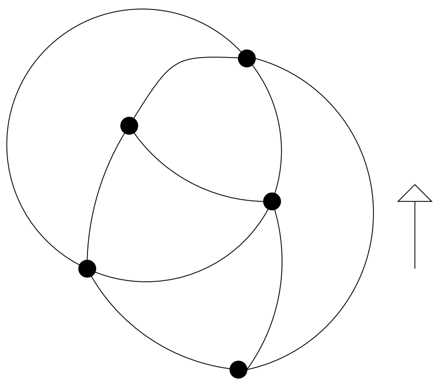
Conjecture [Gessel]

Other nice equations (2)

- The chromatic polynomial of triangulations

$$G(u,v) = \lambda(\lambda-1)v + \lambda^{-1} tuv G(u,1) G(u,v) + tv \frac{G(u,v) - G(0,v)}{u}$$

$$-tuv^2 \frac{G(u,v) - G(u,1)}{v-1}$$



$\lambda(\lambda-1)(\lambda-2)(\lambda-3)^2$

- Solution by Tutte (73-82): let $H(t) = t^2 G(0,1)$; then

$$H''(\lambda t + 10H - 6tH') + \lambda(4-\lambda)(20H + 9t^2 H'' - 18tH') + 2\lambda^2 t(1-\lambda) = 0.$$

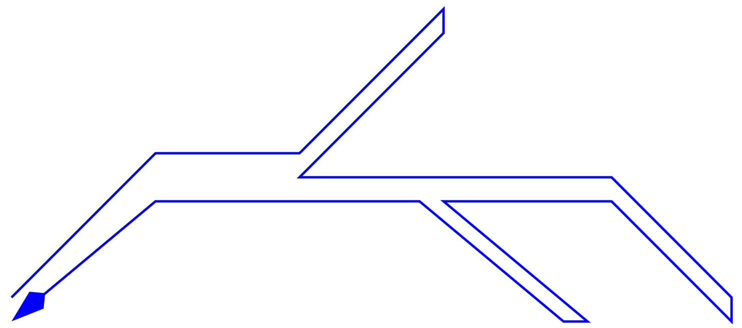
This series is differentially algebraic

Other nice equations (3)

- Discrete positive brownian snake (or maps, or well-labelled trees)

$$S(u,v) = u + tuS(u,1)^2 + tuS(v,1)S(u,1) + tvS(u,1)\,\frac{S(u,v)-S(u,1)}{v-1}.$$

The solution is algebraic



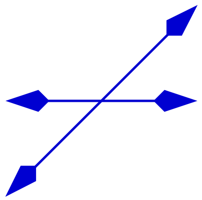
- q-analogue: sum of the height of the steps:

$$S(t;u,v) = u + tuS(t;u,1)^2 + qtuS(tq;v,1)S(t;u,1)$$
$$+tvS(t;u,1)\,\frac{S(t;u,v)-S(t;u,1)}{v-1}.$$

That's it!

• Try to be more systematic (in the linear case? in the linear case when the degree of the kernel is at most 2? cf. [F-I-M])

• ... and first, solve *all* nice examples, in particular



$(1, 0), (1, 1)$
$(-1, 0), (-1, -1)$

• Any other examples?