The Poset Conjecture

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R. Stanley -86 (current version).

Let $(a_i)_{i=0}^d \in \mathbb{R}^{d+1}$. Then (a_i) is said to be

1. <u>*Log-concave*</u> if for all $1 \leq i \leq d-1$

$$a_i^2 \geq a_{i-1}a_{i+1}$$

2. <u>*Unimodal*</u> if there is a $0 \leq k \leq d$ such that

$$a_0 \leq \cdots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \cdots \geq a_d$$

Theorem 1 (Newton). *Let*

$$f(x) = a_0 + a_1 x + \cdots + a_d x^d.$$

Then

$f(x)$ *has real nonpositive zeros* $\Rightarrow$

(a_i) *is log-concave* $\Rightarrow$

(a_i) *is unimodal.*

Let $(P, \leq)$ be a finite partially ordered set.

The *order polynomial*, $\Omega(P;n)$, of $(P, \leq)$ is defined by

$\Omega(P;n) = \#\{\sigma : P \to [n] | x \leq y \Rightarrow \sigma(x) \leq \sigma(y)\}$

If we define

$e_k(P) = \#\{\sigma : P \twoheadrightarrow [k] | \ x \leq y \Rightarrow \sigma(x) \leq \sigma(y)\}$

we see that

$$\Omega(P;n) = \sum_{k=0}^{|P|} e_k(P) \binom{n}{k},$$

so $\Omega(P;n)$ is a polynomial of degree $|P|$ in n.

The E- and W-polynomials associated with P are:

$$
\begin{aligned}
E(P;x) &= \sum_{k=0}^{|P|} e_k(P)x^k \\
\sum_{n\geq 0} \Omega(P;n)x^n &= \frac{W(P;x)}{(1-x)^{|P|+1}}
\end{aligned}
$$

We have the relationship:

$$W(P;x) = (1-x)^{|P|}E(P;\frac{x}{1-x}).$$

Conjecture 1 (Distributive Lattice Conj.). *Let P be a finite poset. Then the polynomials $E(P;x)$ and $W(P;x)$ have real zeros.*

An *order ideal*, I, of a poset P is a subset of P closed under $\leq$, that is

$$x \leq y \wedge y \in I \Rightarrow x \in I.$$

The set of order ideals, $J(P)$, of P is a poset (inclusion).

Theorem 2. *L is a finite distributive lattice if and only if $L = J(P)$ for some P.*

Let $\sigma : P \twoheadrightarrow [k]$ be order preserving. Then

$$\emptyset \subset \sigma^{-1}([1]) \subset \cdots \subset \sigma^{-1}([k-1]) \subset \sigma^{-1}([k]) = P$$

is a chain of length $k+1$ from $\emptyset$ to P in $J(P)$. This relationship is easily seen to be bijective. Hence

$$xE(P;x) = \sum_{c} x^{|c|}$$

where the sum is over all chains from $\emptyset$ to P in $J(P)$.

A stronger conjecture:

Conjecture 2. *The chain polynomial of any modular lattice has only real zeros.*

Let $p = |P|$ and $\omega : P \to [p]$ be any linear extension of P. The *Jordan-Hölder set*, $\mathcal{L}(P)$, of P is the set of permutations $a_1 a_2 \cdots a_p$ such that

$$\omega^{-1}(a_1), \omega^{-1}(a_2), \ldots, \omega^{-1}(a_p),$$

is a linear extension of P.

Theorem 3.

$$W(P; x) = \sum_{\pi \in \mathcal{L}(P)} x^{\mathrm{des}(\pi)+1}$$

The dth *Eulerian polynomial* is

$$A_d(x) = \sum_{\pi \in \mathcal{S}_n} x^{\mathrm{des}(\pi)+1}.$$

Corollary 1 (Euler).

$$\sum_{n \geq 0} n^d x^n = \frac{A_d(x)}{(1-x)^{d+1}}$$

Theorem 4 (Harper). *$A_d(x)$ has only real zeros.*

Operations on Posets

Let P and Q be posets.

1. The <u>*direct sum*</u>, $P \oplus Q$, is the poset on the disjoint union, with $x \leq y$ in $P \oplus Q$ if

   - $x \leq y$ in P or
   - $x \leq y$ in Q or
   - $x \in P$ and $y \in Q$.

2. The <u>*disjoint union*</u>, $P \sqcup Q$, is the poset on the disjoint union, with $x \leq y$ in $P \sqcup Q$ if

   - $x \leq y$ in P or
   - $x \leq y$ in Q.

Theorem 5.

$$xE(P \oplus Q, x) = (x+1)E(P, x)E(Q, x).$$

Corollary 2. *If $E(P; x)$ and $E(Q; x)$ have real zeros, then so does $E(P \oplus Q; x)$.*

Theorem 6 (Wagner). *If $E(P; x)$ and $E(Q; x)$ have real zeros, then so does $E(P \sqcup Q; x)$.*

By a simple counting argument we have:

$$\Omega(P \sqcup Q; x) = \Omega(P; x)\Omega(Q; x).$$

Define $\mathcal{E} : \mathbb{R}[x] \to \mathbb{R}[x]$ by the image of the basis $\binom{x}{k}$, $k = 0, 1, 2, \ldots$:

$$\mathcal{E}\binom{x}{k} = x^k.$$

Then by the definition of the E-polynomial we have

$$\mathcal{E}(\Omega(P; x)) = E(P; x)$$

Define a bilinear function:

$$\diamond : \mathbb{R}[x] \times \mathbb{R}[x] \to \mathbb{R}[x],$$

by

$$f \diamond g = \mathcal{E}\left(\mathcal{E}^{-1}(f)\mathcal{E}^{-1}(g)\right),$$

then

$$E(P \sqcup Q; x) = E(P; x) \diamond E(Q; x).$$

It is easy to see that all real zeros of $E(P; x)$ are in the interval $[-1, 0]$.

Theorem 7 (Wagner). *If $f, g \in \mathbb{R}[x]$ are $[-1, 0]$-rooted, then so is $f \diamond g$.*

The Poset Conjecture is verified for:

1. Column strict labelled Ferrers Posets, i.e. finite order ideals of $\mathbb{N} \times \mathbb{N}$. [**Brenti**].

2. Gaussian Posets [**Brenti**].

3. Disjoint chains [**Simion**].

4. Series Parallel Posets, i.e. posets formed using $\sqcup$ and $\oplus$ [**Wagner**].

5. Forests, i.e. collections of disjoint trees [**Wagner**].

A related result is, [**Gasharov, Skandera**], that the chain polynomial of any $3+1$-free finite poset has real zeros. A poset is $3+1$-free if it does not contain a chain of length 3 and an element incomparable with the elements of this chain.

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