Preface

The central parts of this thesis is shellings of abstract simplicial complexes and positive paths in $\mathbb{Z}^2$.

I will start with a review of some elementary theory of abstract simplicial complexes with a focus on h-vectors of shellable complexes. Then I will discuss a relationship between the reduced Euler characteristic and the genus of the d-cube. After a review of shellable posets I will turn my attention to positive paths in $\mathbb{Z}^2$ and the connection to a certain class of shellable posets, and the 132-avoiding permutations.

I thank my supervisor Einar Steingrímsson for his support, guidance and enthusiasm, and for being more than generous with his time.

Abstract

The thesis begins with a review of some elementary theory of simplicial complexes, in particular shellable complexes and then continues with some aspects of the reduced Euler characteristic and the genus of the d-cube. Here a bijective proof is given proving the fact that the genus of the d-cube is equal to the reduced Euler characteristic of the $(d-2)$-cube. This was first proved indirectly by Beineke and Harary [2], but not in a bijective manner. The second part of the thesis is concerned with positive paths in $\mathbb{Z}^2$ and related numbers such as the ballot, Catalan and Narayana numbers. There is a wealth of literature on the Narayana numbers, concerning various statistics on Dyck paths with a distribution given by the Narayana numbers. A simple method of finding bijections, that prove the equidistribution of some of these statistics, is given. Then it is shown that the Narayana numbers give the h-vectors of the order complexes of a class of posets closely connected to Dyck paths. In this manner it is shown that the number of valleys, the number of high peaks and the number long non-final sequences are equidistributed statistics by showing that each of them is connected to a certain shelling of the order complexes. The thesis ends by relating the statistics on Dyck paths to statistics defined on a certain class of permutations, the 132-avoiding permutations. Here it is shown that the descent statistic on the 132-avoiding permutations has the Narayana distribution.

Contents

1 Introduction

This thesis can by all rights be considered as consisting of at least two separate parts, one concerning the d-cube and one about positive paths. My first aim was to study enough of the theory of simplicial complexes needed in order to find a bijective proof of the fact that the genus of the d-cube is equal to the reduced Euler characteristic of the $(d-2)$-cube, and when this was achieved the study of Björner's [6] theory on shellable posets inspired me to study shellings of the order complex of the Catalan poset of length $2n$ and various statistics defined on Dyck paths. What the two parts thus have in common is that they include methods involving shellings of simplicial complexes.

Recently there has been an increase of influences in combinatorics from other fields of mathematics such as probability theory, algebra and topology. Although the history of geometrical polytopes and geometrical simplicial complexes is long and rich, with names such as Euler, the treatment of abstract simplicial complexes as combinatorial entities is quite new and flourishing. The study of positive paths in $\mathbb{Z}^2$ is also rich and has been treated by different schools such as statistics, probability theory, formal languages, physics and of course combinatorics.

There are very few graphs whose genus is known. In 1963 Beineke and Harary [2] proved that whenever $d \geq 2$, the genus of the d-cube is given by,

$$\gamma(C^d) = (d-4)2^{d-3} + 1.$$

It is easy to show that the reduced Euler characteristic of the d-cube is given by

$$\tilde{\chi}(C^d) = (d-2)2^{d-1} + 1.$$

The genus of the d-cube is consequently equal to the reduced Euler characteristic of the $(d-2)$-cube. A natural question is if this relationship can be proved bijectively, that is, if one can find a bijection that associates a "handle" in the embedding of the d-cube to a unique facet (that shares two vertices with preceding facets) in a shelling of the $(d-2)$-cube. Using Beineke and Harary's construction in [2], and a particular shelling, a bijective proof of this relationship is given.

There is a wealth of literature concerning statistics defined on Dyck paths with a distribution given by the Narayana numbers. We a give bijective proofs of the equidistribution of some of these statistics by concidering a class of recursively defined bijections. It is also shown that the h-vector of the order complex of a class of posets closely connected to Dyck paths equals the Narayana numbers. By studying different shellings of these order complexes it is shown that the number of valleys, the number of high peaks and the number of long non final sequences are equidistributed statistics. Finally, we discuss the connection between Dyck paths and the 132-avoiding permutations, and show that the distribution of the descent statistic on this class of permutations is given by the Narayana numbers.

2 Simplicial complexes and shellings

Here we will review some of the theory concerning shellings of simplicial complexes and h-vectors, which can be found in [13].

Definition 2.1. An (abstract) *simplicial complex* Δ on a vertex set V is a collection of subsets F of V satisfying:

(i). if $x \in V$ then $\{x\} \in \Delta$,

(ii). if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$.

The elements of Δ are called *faces*, and if $|F| = k + 1$, then F is called a *k-face*. We say that a k-face has dimension k, and the dimension of Δ is the largest dimension of its faces. A maximal face (with respect to inclusion) is called a *facet*, and a simplicial complex is said to be *pure* if all its facets have the same dimension. The empty set is considered a face of dimension -1.

There are two vectors of topological and combinatorial interest associated with a simplicial complex, namely:

Definition 2.2. The *f-vector* of a simplicial complex Δ of dimension $(d-1)$ is $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$, where f_k is the number of k-faces in Δ. The *h-vector*, $h(\Delta) = (h_0, h_1, \ldots, h_d)$ is defined in terms of the f-vector by:

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i}, \tag{1}$$

where $f_{-1} = 1$.

The following relationships follow immediately from the definition of the h-vector:

$$h_1 = f_0 - d = \#V - d, \tag{2}$$

$$f_{d-1} = h_0 + h_1 + \cdots + h_d, \tag{3}$$

$$h_d = (-1)^{d-1} \sum_{i=-1}^{d-1} (-1)^i f_i = (-1)^{d-1}\tilde{\chi}(\Delta), \tag{4}$$

where the *reduced Euler characteristic* $\tilde{\chi}(\Delta)$ of a simplicial complex Δ is an important topological invariant.

Definition 2.3. A pure simplicial complex Δ of dimension $d-1$ is *shellable* if its facets can be ordered $F_1, F_2, \dots, F_n$ in such a way that any of the conditions below hold. We will show later that these conditions indeed are equivalent. Let $\Delta_i = \bigcup_{j \leq i} 2^{F_j}$, where $2^F = \{G \mid G \subseteq F\}$.

(i). For all $i \in [n] = \{1, 2, \dots, n\}$, $\Delta_i \setminus \Delta_{i-1} (= 2^{F_i} \setminus \Delta_{i-1})$ has a unique minimal element with respect to inclusion. We call the unique minimal element the *restriction* $r(F_i)$ of F_i.

(ii). For all i, $2 \leq i \leq n$, $\Delta_{i-1} \cap 2^{F_i}$ is pure of dimension $d-2$.

(iii). For all k and i such that $1 \leq k < i \leq n$ there is a j such that $1 \leq j < i$ and

$$F_i \cap F_k \subseteq F_i \cap F_j \text{ and } F_i \cap F_j \text{ is of dimension } d-2.$$

For sets S and T define the (closed) *interval* $[S, T]$ by $[S, T] = \{A \mid S \subseteq A \subseteq T\}$. That $2^{F_i} \setminus \Delta_{i-1}$ has a unique minimal element $r(F_i)$ is equivalent to that $2^{F_i} \setminus \Delta_{i-1} = [r(F_i), F_i]$, since $H \in [r(F_i), F_i]$ and $H \in 2^{F_i} \cap \Delta_{i-1}$ would imply $r(F_i) \in 2^{F_i} \cap \Delta_{i-1}$. If the first condition holds we know that Δ can be written as a disjoint union of intervals

$$\Delta = [r(F_1), F_1] \cup \dots \cup [r(F_n), F_n].$$

This leads us to the following definition.

Definition 2.4. A pure simplicial complex Δ is *partitionable* if Δ can be written as a disjoint union

$$\Delta = [G_1, F_1] \cup \dots \cup [G_n, F_n], \tag{5}$$

where each F_i is a facet of Δ. The right-hand side of (5) is a *partitioning* of Δ.

That the conditions given in Definition 2.3 indeed are equivalent is shown below.

Lemma 2.5. *The conditions for a sequence $F_1, F_2, \dots, F_n$ of facets of a pure simplicial complex Δ given in Definition 2.3 are equivalent.*

Proof. To each facet G_j, $1 \leq j \leq s$, in $\Delta_{i-1} \cap 2^{F_i}$ associate the vertex v_j such that $G_j \cup \{v_j\} = F_i$. Then it is clear that $G = \{v_1, v_2, \dots, v_s\}$ is the unique minimal element since $2^{F_i} \setminus \Delta_{i-1} = [G, F_i]$.

On the other hand, if $i \neq 1$ then $r(F_i) \neq \emptyset$. Since $2^{F_i} \setminus \Delta_{i-1} = [r(F_i), F_i]$, if $H \in \Delta_{i-1} \cap 2^{F_i}$ then $H \subseteq F_i \setminus \{v_j\}$ for some $v_j \in r(F_i)$. Since it is obvious that conditions 2 and 3 are equivalent this establishes the Lemma. $\square$

Proposition 2.6. *Let* (5) *be a partitioning of* Δ. *Then*

$$h_i = \#\{j : |G_j| = i\}. \tag{6}$$

Equivalently,

$$\sum_{i=0}^{d} h_i x^i = \sum_{j=1}^{n} x^{|G_j|}. \tag{7}$$

Proof. The left-hand side of (1) may be expressed as $\sum_{F\in\Delta}(x-1)^{d-|F|}$. Since we have a partitioning given, we have

$$\sum_{i=0}^{d} h_i x^{d-i} = \sum_{j=1}^{n} \sum_{G_j\subseteq F\subseteq F_j} (x-1)^{d-|F|}.$$

If $|G_j| = i$ then, since $|F_j| = d$, we have

$$\sum_{G_j\subseteq F\subseteq F_j} (x-1)^{d-|F|} = \sum_{A\subseteq[d-i]} (x-1)^{d-(i+|A|)} =$$

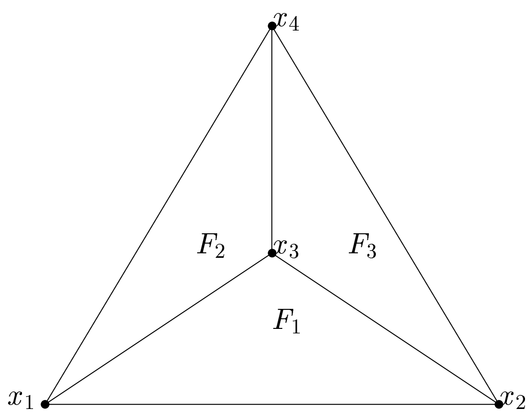
$$= \sum_{j=0}^{d-i} \binom{d-i}{j} (x-1)^{d-i-j} = x^{d-i} = x^{d-|G_j|},$$

and the proof is complete. □

Since $\#r(F_i)$ equals the number of facets in $\Delta_{i-1}\cap 2^{F_i}$ this gives us a way of computing the h-vector for a shellable simplicial complex. This turns out to be useful.

Example 2.7. Let Δ be the simplicial complex on the vertex set $V = \{x_1, x_2, x_3, x_4\}$ with facets $F_1 = \{x_1, x_2, x_3\}$, $F_2 = \{x_1, x_3, x_4\}$ and $F_3 = \{x_1, x_3, x_4\}$.

Then F_1, F_2, F_3 is a shelling. We have that 2^{F_2} intersects 2^{F_1} in one two-element set, and 2^{F_3} intersects $2^{F_1}\cup 2^{F_2}$ in two two-element sets, hence $h(\Delta) = (1, 1, 1, 0)$ according to (6).



3 The d-cube

The *d-cube* will in this section be looked upon as a simplicial complex and as a graph. When the genus of the d-cube is discussed we consider the d-cube as a graph, and when the h-vector is in question the d-cube is looked upon as a simplicial complex. Since the graph and the simplicial complex are essentially the same, this amounts to just a slight abuse of notation.

Definition 3.1. The objects below we will refer to as the d-cube C^d:

- When looked upon as a simplicial complex of dimension 1 the d-cube has vertex set $V(C^d) = \{0,1\}^d$ and facets

$$\{\{v,w\} \mid v,w \in V(C^d), |v-w| = 1\}.$$

  Here $|\cdot|$ is the standard Euclidean norm. The condition that the vertices that constitute a facet are separated by distance one is equivalent to the condition that these vertices differ in exactly one coordinate, when written in binary form.

- The graph C^d has the same vertex set as above and the edges are between the vertices that make up a facet in the simplicial complex.

I will, for convenience, denote facets and edges by words, or as functions

$$F(x) = a_1 a_2 \cdots a_{k-1} x a_{k+1} \cdots a_d,$$

where $a_i \in \{0,1\}$ and x is a variable with values in $\{0,1\}$. Written in this way the facet/edge F has vertices $F(0)$ and $F(1)$. The number of facets/edges in C^d is consequently $d2^{d-1}$.

3.1 The h-vector of the d-cube

If $\{F_i\}_{i=1}^n$ is an ordering of the facets of a simplicial complex Δ of dimension one then $\{F_i\}_{i=1}^n$ is a shelling if and only if for each $i \neq 1$ we have

$$F_i \cap \bigcup_{j<i} F_j \neq \emptyset.$$

This follows from Definition 2.3.(iii). since the intersection of two different facets is either empty or a singleton set. A simplicial complex of dimension one is thus shellable if and only if it is connected and since the d-cube clearly is connected it is shellable. In order to establish a connection between the genus and the h-vector of C^d we will study a particular shelling, namely a certain *lexicographical* shelling of C^d. It turns out that lexicographical orders often are a way to find shellings, and we will show that the facets of C^d allow such a shelling. If we have an order $<$ on a set M, we define the *lexicographic* order $<_L$ on M^d to be the following: If $A, B \in M^d$ and $A = (a_1, \dots, a_d)$, $B = (b_1, \dots, b_d)$ then

$$A <_L B \text{ iff } a_k < b_k \text{ for } k = min\{i \mid a_i \neq b_i\}.$$

It is clear that if $<$ is a linear order then so is $<_L$.

Lemma 3.2. *Let $0 < x < 1$ determine a linear order on $\{0, x, 1\}$. If we order the facets of C^d by the lexicographical order on $\{0, x, 1\}^d$, we have a shelling $F_1, F_2, \dots, F_{d2^{d-1}}$ such that for all $i > 1$,*

$$\#r(F_i) = 2 \quad \Leftrightarrow$$

$$F_i = a_1 a_2 \cdots a_{k-1} x a_{k+1} \cdots a_d \textit{ and } a_i = 1 \textit{ for some } 1 \leq i \leq k-1.$$

Proof. Let $i > 1$. All facets intersecting $F_i = a_1 a_2 \cdots a_{k-1} x a_{k+1} \cdots a_d$ maximally can be obtained from F_i by replacing x with a 0 or a 1 and moving x to any of the $d-1$ other places in the word. If we move x to the right we have to replace x with a 0 to get a smaller facet. If we move x to the left we have to move it to a place where there is a 1 to get a smaller facet, and we can then replace x with either 0 or 1. If $k < d$ we can move x to the right and if $k = d$ then (since $i > 1$) F_i has a 1 to the left of its x, hence we can move x to the left to get a smaller facet. We have shown that $F_i(0) \in F_i \cap \bigcup_{j<i} F_j$, thus $\{F_i\}_{i=1}^n$ is a shelling, and

$$F_i(1) \in F_i \cap \bigcup_{j<i} F_j \text{ if and only if the } x \text{ in } F_i \text{ has a 1 to its left.}$$

□

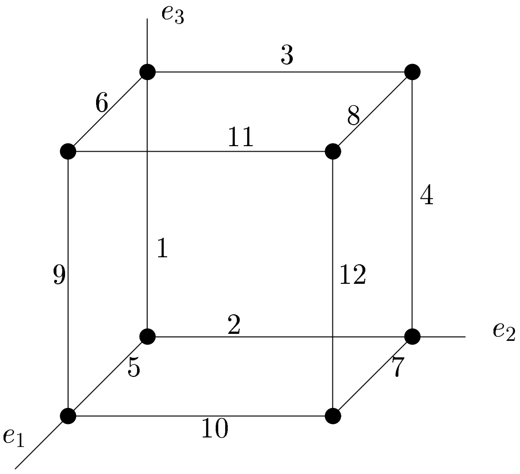
Example 3.3. The lexicographical shelling of C^3 is the following:

$$\begin{aligned} F_1 &= 00x, \quad F_2 = 0x0, \quad F_3 = 0x1, \\ F_4 &= 01x, \quad F_5 = x00, \quad F_6 = x01, \\ F_7 &= x10, \quad F_8 = x11, \quad F_9 = 10x, \\ F_{10} &= 1x0, \quad F_{11} = 1x1, \quad F_{12} = 11x. \end{aligned}$$

The facets sharing both their vertices with facets before them in the ordering are

$$F_4, \quad F_9, \quad F_{10}, \quad F_{11}, \quad F_{12},$$

hence $h_2(C^3) = 5$.



We now have a complicated way of computing $h_2(C^d)$. According to (6) and Lemma 3.2, we have:

$$\begin{aligned} h_2(C^d) &= \#\{a_1 \cdots a_{k-1} x a_{k+1} \cdots a_d \mid a_i = 1 \text{ for some } 1 \leq i \leq k-1\} = \\ &= \sum_{k=1}^{d} 2^{d-k}(2^{k-1}-1) = (d-2)2^{d-1}+1. \end{aligned}$$

Since $h_0(C^d) = 1$ we have $h_1(C^d) = d2^{d-1} - (d-2)2^{d-1} - 1 - 1 = 2^d - 2$. An easier way to compute $h(C^d)$ is from (4), since the h-vector only has two non-trivial coordinates.

3.2 The genus of the d-cube

In this paper it will suffice with a rather naive concept of genus. I will not give a formal definition of the genus of an orientable surface, because we will limit ourselves to surfaces where this is obvious. We say that an orientable surface with n *handles* has genus n.

Definition 3.4. The *genus* $\gamma(G)$ of a graph G is the minimal genus among all orientable surfaces in which the graph can be drawn without intersection of its edges.

The following Lemma is due to Beineke and Harary [3], and is stated without a proof:

Lemma 3.5. *Whenever $d \geq 2$, the genus of the d-cube satisfies*

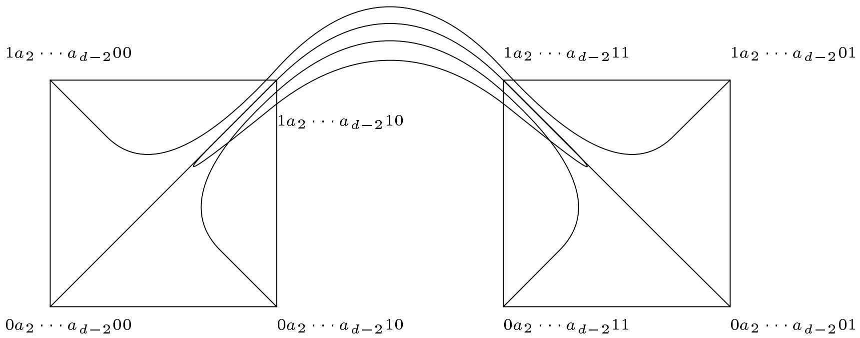
$$\gamma(C^d) \geq h_2(C^{d-2})$$

Let the *quadruples* of C^d be the subgraphs of four vertices such that each vertex has an edge to exactly two other vertices in the quadruple. I will refer to the quadruples as words

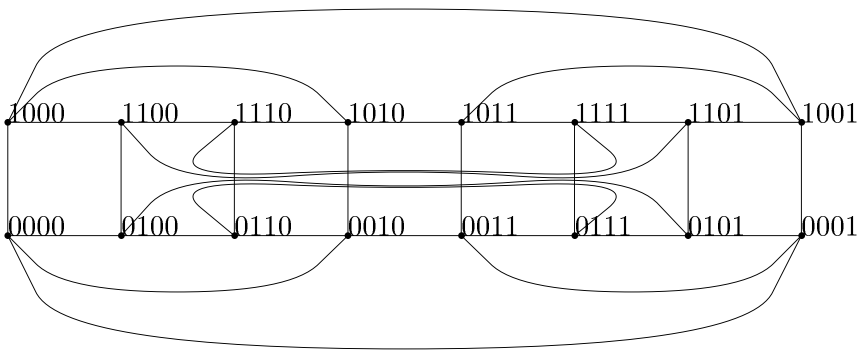
$$a_1 \cdots a_{\ell-1} x a_{\ell+1} \cdots a_{k-1} y a_{k+1} \cdots a_d,$$

given the obvious interpretation. If we can embed the graph C^d on a surface with $h_2(C^{d-2})$ handles then we have $\gamma(C^d) \leq h_2(C^{d-2})$.

The following embedding is due to Beineke and Harary [2]. We want to construct an embedding of C^d inductively as follows. Assume that we have embedded C^{d-1} in the orientable surface S_{d-1}. Place the mirror image $\tilde{S}_{d-1}$ of S_{d-1} to the right of S_{d-1} (in the plane). Suffix a 0 to each vertex in S_{d-1} and a 1 to each vertex in $\tilde{S}_{d-1}$. Now, place a handle (and join opposite vertices) from each quadruple $xa_2 \cdots a_{d-2}y0$ in S_{d-1} to the quadruple $xa_2 \cdots a_{d-2}y1$ in $\tilde{S}_{d-1}$, except for the quadruples $x0 \cdots 0ya_d$ which we can join without handles. (See the figure below describing how a handle is placed).



If this construction is possible, it is clear that all vertices will be joined, since the only vertices that were not joined before we added the new handles were the ones differing in the last place, and the new handles make it possible to join these, hence we have an embedding of the d-cube. We will show that this construction is possible by induction. The construction is illustrated here in detail for $d = 4$:

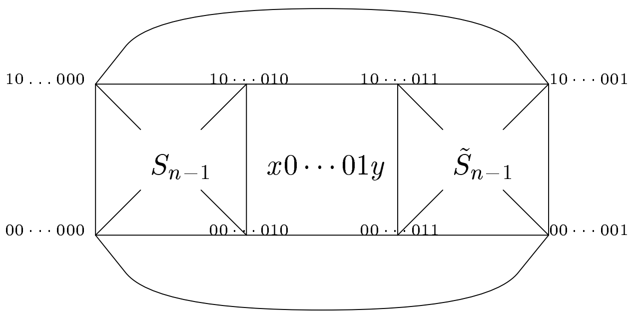


Lemma 3.6 (Beineke and Harary [2]). *The d-cube C^d can be embedded in the orientable surface S_d in the way described above.*

Proof. Assume that we have embedded C^n for some $n \geq 4$ in the way described. We want to place a handle on each quadruple

$$xa_2 \cdots a_{n-1}y \text{ where } a_i = 1 \text{ for some } i$$

In other words, we require the quadruple to constitute a "face". If $a_2 \cdots a_{n-2} = 00 \cdots 0$ then $x00 \cdots 01y$ is one of the quadruples whose vertices were joined without a handle, and this clearly constitutes a face (see figure below). If $a_2 \cdots a_{n-2} \neq 00 \cdots 0$, then we were able to place a handle from $xa_2 \cdots a_{n-2}y0$ to $xa_2 \cdots a_{n-2}y1$ in the preceding step. Then we can place a handle on $xa_2 \cdots a_{n-1}y$ (see figure above). Since the quadruples $x0 \cdots 0ya_{n+1}$, where $a_{n+1} \in \{0, 1\}$ clearly can be established without using handles we are able to embed C^{n+1} in the way described, and the Lemma is proved. □



We are now ready to prove that the genus of the d-cube is equal to the reduced Euler characteristic of the $(d-2)$-cube. This is shown by identifying each handle in the surface S_d with a unique facet F in C^{d-2} such that $\#r(F) = 2$ in the lexicographical shelling of C_{d-2}.

Theorem 3.7. *Whenever $d \geq 2$, the genus of the d-cube is given by*

$$\gamma(C^d) = h_2(C^{d-2}) \tag{8}$$

Proof. In the embedding of C^k, $k \leq n$, handles were placed from each quadruple $xa_2 \cdots a_{k-2}y0$ in S^{k-1} to the corresponding quadruple $xa_2 \cdots a_{k-2}y1$ in $\tilde{S}_{k-1}$ such that $a_i = 1$ for some $2 \leq i \leq k-2$. When we proceed with the embedding we will have "copies" of this handle between the quadruples $xa_2 \cdots a_{k-2}y0a_{k+1} \cdots a_d$ and $xa_2 \cdots a_{k-2}y1a_{k+1} \cdots a_d$ for all $a_{k+1} \cdots a_d \in \{0, 1\}^{d-k}$. Removing the x and the integer after the y we see that the handles in C^d can uniquely be described by all

$$b_1 \cdots b_{k-1}yb_{k+1} \cdots b_{d-2} \text{ where } b_i = 1 \text{ for some } 1 \leq i \leq k-1$$

which according to Lemma 3.2 are the facets F in C^{d-2} such that $\#r(F) = 2$ in the lexicographical shelling of C^{d-2}. Hence we have a bijection between the handles of S_d and the facets F in C^{d-2} such that $\#r(F) = 2$ and the Theorem is proved. □

4 Lexicographically shellable posets

There is a certain kind of simplicial complexes where we can find a sufficient condition for shellability. These simplicial complexes are constructed from posets (partially ordered sets). A *chain* C in a poset $(P, \leq)$ is a subset of P such that the induced order $\leq$ is linear on C. An *unrefinable* chain is a chain such that if x and y are elements of the chain there is no z not in the chain such that $x \leq z \leq y$. A poset is *bounded* if it has a least element and a greatest element. A finite poset is said to be *pure* if all maximal chains have the same number of elements and *graded* if it is bounded and pure. The *length* of a pure poset is the number of elements in a maximal chain minus one. We say that an element y *covers* x, denoted $x \prec y$, if $x < y$ and $x < z < y$ for no z.

Definition 4.1. The *order complex* $\Delta(P)$ of a finite poset P is the simplicial complex of all chains of P. A finite pure poset P is said to be *shellable* if $\Delta(P)$ is shellable.

We will review some of the theory concerning order complexes and shellability that can be found in [6].

Definition 4.2. Let P be a graded poset and $C(P) = \{(x, y) \in P^2 \mid x \prec y\}$ its covering relations.

(i). An *edge-labeling* of P is a map $\lambda : C(P) \to \Lambda$, where Λ is any poset. An unrefinable chain $x_0 \prec x_1 \prec \cdots \prec x_n$ is called *rising* if

$$\lambda(x_0, x_1) \leq \lambda(x_1, x_2) \leq \cdots \leq \lambda(x_{n-1}, x_n).$$

I will also use *rising* as a noun referring to a rising unrefinable chain.

(ii). An *R-labeling* is an edge-labeling λ such that for each interval $[x, y]$ there is a unique rising unrefinable chain $x = x_0 \prec x_1 \prec \cdots \prec x_n = y$.

(iii). An R-labeling λ is said to be an *L-labeling* if for any interval $[x, y]$, if

$$x = x_0 \prec x_1 \prec \cdots \prec x_n = y$$

is the unique rising unrefinable chain then $\lambda(x, x_1) < \lambda(x, z)$ for all $x \prec z \leq y$, $z \neq x_1$.

Definition 4.3. A poset is *lexicographically shellable* (or *L-shellable*) if it is graded and admits an L-labeling.

It will be proved that lexicographically shellable posets are indeed shellable but first we need some useful results. To each unrefinable n-chain $\mathbf{c} : x_0 \prec x_1 \prec \cdots \prec x_n$ of a poset P with an R-labeling $\lambda : C(P) \to \Lambda$ associate the n-tuple

$$\pi(\mathbf{c}) = (\lambda(x_0, x_1), \lambda(x_1, x_2), \ldots, \lambda(x_{n-1}, x_n)) \in \Lambda^n,$$

often referred to as the *Jordan-Hölder sequence* of $\mathbf{c}$. For each interval $[x,y]$ in P, with an R-labeling λ, let $\mathbf{c}(x,y)$ denote the unique rising from x to y.

Lemma 4.4. *Let $\lambda : C(P) \to \Lambda$ be an R-labeling of a graded poset P. Then the following conditions are equivalent:*

(i). *λ is an edge-labeling.*

(ii). *For each interval $[x,y]$ in P and every unrefinable chain $\mathbf{d}$ from x to y other than $\mathbf{c}(x,y)$ it is true that $\pi(\mathbf{c}(x,y)) < \pi(\mathbf{d})$ in the lexicographical order.*

Proof. Let $\mathbf{c}(x,y) : x = x_0 \prec x_1 \prec \cdots \prec x_n = y$ be the unique rising in $[x,y]$.

(i). $\Rightarrow$ *(ii).* : For $\mathbf{d} : x = y_0 \prec y_1 \prec \cdots \prec y_n = y$, an unrefinable chain in $[x,y]$, let k be the least index such that $x_k \neq y_k$. Then $x_{k-1} \prec \cdots \prec x_n = y$ is the unique rising in $[x_{k-1}, y]$ and since λ is an L-labeling we have $\lambda(x_{k-1}, x_k) < \lambda(y_{k-1}, y_k)$ which implies $\pi(\mathbf{c}(x,y)) < \pi(\mathbf{d})$

(ii). $\Rightarrow$ *(i).* : If $x \prec z_1 \leq y$ and $z_1 \neq x_1$ then $\lambda(x,x_1) \leq \lambda(x,z_1)$. Let $z_1 \prec z_2 \prec \cdots \prec z_n = y$ be the unique rising in $[z_1, y]$. If $\lambda(x,z_1) = \lambda(x,x_1)$ then we have

$$\lambda(x,z_1) = \lambda(x,x_1) \leq \lambda(x_1,x_2) \leq \lambda(z_1,z_2) \leq \cdots \leq \lambda(z_{n-1},y)$$

contradicting the uniqueness of $\mathbf{c}(x,y)$. Hence $\lambda(x,z_1) < \lambda(x,x_1)$. $\square$

For examples of L-labelings see Section 6 where a couple of L-labelings are given. If Λ is a poset we say that a sequence $\pi = (a_1, a_2, \ldots, a_n) \in \Lambda^n$ has a *descent* at i, $i \in [n-1]$, if $a_i \not\leq a_{i+1}$. The sequence π is said to have the *descent set* $D(\pi) = \{i \in [n-1] \mid a_i \not\leq a_{i+1}\}$.

Lemma 4.5. *Let $\lambda : C(P) \to \Lambda$ be an L-labeling of a graded poset P of length n. Then the following conditions for a maximal chain $\mathbf{m} : \hat{0} = x_0 \prec x_1 \prec \cdots \prec x_n = \hat{1}$ are equivalent:*

- *$\pi(\mathbf{m})$ has a descent at $e \in [n-1]$.*
- *$\mathbf{m} \cap \mathbf{h} = \mathbf{m} \setminus \{x_e\}$ for some maximal chain $\mathbf{h}$ such that $\pi(\mathbf{h}) < \pi(\mathbf{m})$ in the lexicographical order on Δ^n.*

Proof. Let $x_{e-1} \prec y \prec x_{e+1}$ be the unique rising in $[x_{e-1}, x_{e+1}]$. Thus $\pi(\mathbf{m})$ has a non-descent at e iff $x_e = y$ iff there is no maximal chain $\mathbf{h}$ such that $\pi(\mathbf{h}) < \pi(\mathbf{m})$ in the lexicographical order on Δ^n and $\mathbf{m} \cap \mathbf{h} = \mathbf{m} \setminus \{x_e\}$ (since λ is an L-labeling). $\square$

We are now prepared to prove the central theorem of this section. Let $\mathcal{M}$ denote the set of all maximal chains of a poset P.

Theorem 4.6. *If P is lexicographically shellable then P is shellable.*

Proof. Let Ω be any linear order on $\mathcal{M}$ such that $\pi(\mathbf{k}) < \pi(\mathbf{m})$ implies $\mathbf{k} <_\Omega \mathbf{m}$. Such an order always exist. We want to prove

$$\mathbf{k} <_\Omega \mathbf{m} \Rightarrow \mathbf{k} \cap \mathbf{m} \subseteq \mathbf{h} \cap \mathbf{m} = \mathbf{m} \setminus \{x\} \text{ for some } \mathbf{h} <_\Omega \mathbf{m} \text{ and } x \in \mathbf{m}$$

Let $\mathbf{m} : \hat{0} = x_0 \prec x_1 \prec \cdots \prec x_n = \hat{1}$ and $\mathbf{k} : \hat{0} = y_0 \prec y_1 \prec \cdots \prec y_n = \hat{1}$ be two maximal unrefinable chains in P. Suppose $\mathbf{k} <_\Omega \mathbf{m}$. Let r be the least index such that $x_r \neq y_r$, and s the least index $s > r$ such that $x_s = y_s$, then $x_{r-1} \prec x_r \prec \cdots \prec x_s$ is not the unique rising in $[x_{r-1}, x_s]$, since then we would have $\pi(\mathbf{m}) < \pi(\mathbf{k})$. Since $\pi(\mathbf{m})$ thus has a descent at some $e \in [r, s]$ it follows by Lemma 4.5 that $\mathbf{k} \cap \mathbf{m} \subseteq \mathbf{h} \cap \mathbf{m} = \mathbf{m} \setminus \{x_e\}$ for some $\mathbf{h} <_\Omega \mathbf{m}$, and $\Delta(P)$ is shellable. ☐

Corollary 4.7. *If P (of length n) allows an L-labeling $\lambda : C(P) \to \Lambda$ such that the set $\{\pi(\mathbf{m}) \mid \mathbf{m} \in \mathcal{M}\}$ is linear with respect to the lexicographical order on Λ^n and $\pi(\mathbf{m}) = \pi(\mathbf{h})$ for no $\mathbf{m} \neq \mathbf{h}$ then the h-vector of $\Delta(P)$ satisfies*

$$h_k(\Delta(P)) = |\{\mathbf{m} \in \mathcal{M} \mid \#D(\pi(\mathbf{m})) = k\}|.$$

Proof. If $\{\pi(\mathbf{m}) \mid \mathbf{m} \in \mathcal{M}\}$ is linear with respect to the lexicographical order on Λ^n and $\pi(\mathbf{m}) = \pi(\mathbf{h})$ for no $\mathbf{m} \neq \mathbf{h}$ then we can use the lexicographical order on Λ^n for Ω (in the proof of Theorem 4.6) . The Corollary now follows from Lemma 4.5 and (7). ☐

5 Paths, Ballots and Narayana numbers

In this section we will review some classic work in combinatorics and probability theory concerning positive paths in $\mathbb{Z}^2$. The Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

are members of a much studied family of numbers called the *ballot numbers*, $b(n, k)$, that count the number of different elections possible between two candidates A and B, where the votes are counted one at a time, such that during the vote A always has a greater or equal number of votes and the final number of votes for A and B are n and k. The ballot problem, to find $b(n, k)$, can be traced back to 1887 when it was stated and solved by J. Bertrand [5]. In order to give a solution to the ballot problem and related problems we need some definitions.

Definition 5.1. A *path* is a word $w = a_1 a_2 \cdots a_n$ on the alphabet $\{u, d\}$ where $u = (1, 1)$ and $d = (1, -1)$ are called *ascents* and *descents*. To w we associate a sequence of points $P_0 P_1 \cdots P_n$ where

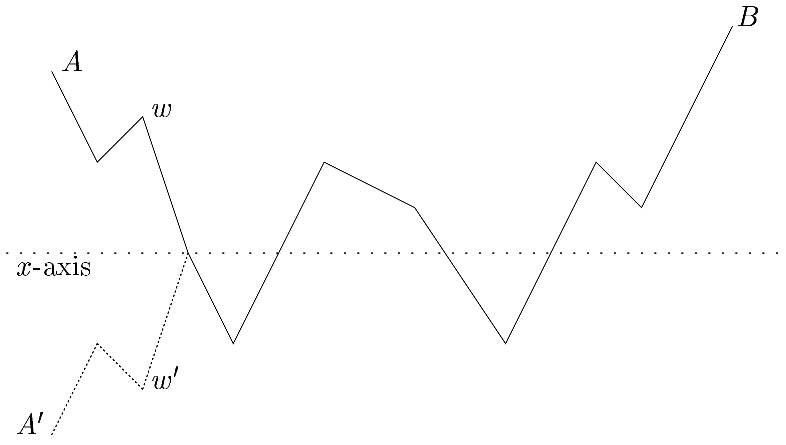
(i). $P_0 \in \mathbb{Z}^2$ and

(ii). $P_k = P_{k-1} + a_k$ for $k \in [n]$.

The empty word, denoted 1, is also considered a path. The ith *height* $h_i(w)$ of a path w is the y-coordinate of the corresponding point P_i. The *positive* paths are the sequences $w = a_1 \cdots a_n$ such that $P_0 P_1 \cdots P_n$ never goes below the x-axes, and the *Dyck* paths are the positive paths such that $P_0 = P_n = (0,0)$. For a path w and $x \in \{u, d\}$ let $|w|_x$ be the number of x's in w, and let the length $|w|$ of a Dyck path w be $|w|_u$. Let $\mathcal{C}_n$ denote the set of all Dyck paths of length n and let $\mathcal{C}_\infty = \cup_{n \geq 0} \mathcal{C}_n$ be the set of all Dyck paths. Thus $b(n,k)$ is the number of positive paths w such that $|w|_u = n$ and $|w|_d = k$. The Catalan number C_n is defined to be $b(n,n)$. The classical proof of the Ballot theorem makes use of a simple and elegant Lemma attributed to D. André [1].

Lemma 5.2 (Reflection principle). *The number of paths from $A = (0, h_0)$ to $B = (n, h_n)$, such that $0 < h_0$ and $0 \leq h_n$, which touch or cross the x-axes (i.e $h_k \leq 0$ for some $k \in [0,n]$) equals the number of paths from $A' = (0, -h_0)$ to B.*

Proof. Let $w = h_0 h_1 \cdots h_n$ be the heights of a path from A to B with $h_k \leq 0$ for some $k \in [0,n]$ and let t be the least index such that $h_t = 0$. Define the reflection w' of w to be the path with heights $(-h_0)(-h_1) \cdots (-h_t) h_{t+1} \cdots h_n$. This clearly establishes a one-to-one correspondence between the paths from A to B which touch/cross the x-axis and the paths from A' to B. $\square$



The Ballot Theorem now follows as a Corollary to the Reflection principle.

Theorem 5.3 (The Ballot Theorem). *For all natural numbers n and m we have*

$$b(n,m) = \frac{n+1-m}{n+1}\binom{n+m}{n} = \binom{n+m}{n} - \binom{n+m}{n+1}$$

Proof. The total number of paths with n ascents and m descents is $\binom{n+m}{n}$, since we want n places in the word to be ascents. The Theorem will follow if we prove that there are $\binom{n+m}{n+1}$ non- positive paths between $(0,0)$ and $(n+m, n-m)$. By a change of axis this is the number of paths between $(0,1)$ and $(n+m, n-m+1)$ that cross or touch the x-axis, which by the Reflection principle is equal to the number of paths between $(0,-1)$ and $(n+m, n-m+1)$. By a change of axis we have that this is the number of paths between $(0,0)$ and $(n+1+m-1, n+1-(m-1))$ which in turn is $\binom{n+m}{n+1}$. □

There is a refinement of the Catalan numbers that is frequent in the literature which we now will be acquainted with, namely the *Narayana* numbers first studied by T. V. Narayana [10],

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}.$$

That $N(n,k)$ is a refinement of the Catalan numbers means:

$$\sum_k N(n,k) = C_n.$$

First a couple of definitions.

Definition 5.4. A statistic σ on a set M is any mapping $\sigma : M \to \mathbb{N}$. The distribution $D_\sigma : \mathbb{N} \to \mathbb{N}$ of σ is defined by

$$D_\sigma(k) = |\sigma^{-1}(\{k\})|.$$

If $\Sigma : \sigma_1, \sigma_2, \dots, \sigma_n$ is a sequence of statistics on M, then the joint distribution, $D_\Sigma : \mathbb{N}^n \to \mathbb{N}$, of Σ is defined by

$$D_\Sigma(k_1, k_2, \dots, k_n) = |\sigma^{-1}(\{(k_1, k_2, \dots, k_n)\})|,$$

where σ is the mapping

$$x \mapsto (\sigma_1(x), \sigma_2(x), \dots, \sigma_n(x)).$$

We will show that the following statistics on $\mathcal{C}_\infty$ have a distribution given by the Narayana numbers, when considered as statistics defined on $\mathcal{C}_n$. Let w be a path in $\mathcal{C}_n$, then

- $\mathrm{P}(w)$ is the number of *peaks* (sequences ud) in w,
- $\mathrm{HP}(w)$ is the number of *high peaks*, that is peaks such that the u in ud is at a height greater than one,
- $\mathrm{V}(w)$ is the number of *valleys* (sequences du) in w,

- $\mathrm{DA}(w)$ is the number of *double ascents* (sequences uu) in w,
- $\mathrm{EA}(w)$ is the number of *even ascents*, that is the number of ascents at even height in w,
- $\mathrm{OA}(w)$ is the number of *odd ascents*, that is the the number of ascents at odd height in w.
- $\mathrm{LNFS}(w)$ is the number of *long non-final sequences*, more precisely the number of sequences uud and ddu in w.

In order to show some facts about these statistics we need a Lemma. For all $d \in [0, n-1]$ let π_d denote the cyclic permutation $(d+1, d+2, \dots, n, 1, \dots, d) \in \mathcal{S}_n$.

Lemma 5.5. *Let* $(a_1, a_2, \dots, a_n) \in \mathbb{Z}^n$ *be such that* $\sum_{i=1}^n a_i = 1$. *Then there is a unique permutation* $\pi_d \in \mathcal{S}_n$ *such that* $\sum_{i=1}^k a_{\pi_d(i)} \geq 1$ *for all* $k \in [n]$.

Proof. Existence : Let d be the maximal index m such that $\sum_{i=1}^m a_i$ is minimal. Now $\sum_{i=d+1}^j a_i \geq 1$ for all $d+1 \leq j \leq n$, and for $0 \leq j \leq d$ we have

$$\sum_{i=d+1}^{k} a_i + \sum_{i=1}^{j} a_i \geq (1 - \sum_{i=1}^{d} a_i) + \sum_{i=1}^{d} a_i = 1.$$

The permutation π_d has the desired properties.

Uniqueness : If π_c and π_d both satisfy the conditions we can assume that $c = 0$ (let $b_i = a_{\pi_c(i)}$, then $a_{\pi_d(i)} = b_{\pi_c^{-1} \circ \pi_d(i)}$). But then $\sum_{i=1}^n a_i = \sum_{i=1}^d a_i + \sum_{i=d+1}^n a_i \geq 1 + 1 = 2$ which is a contradiction. $\square$

Definition 5.6. A *composition of* n *into* k *parts* is a vector $(a_1, a_2, \dots, a_k)$ of positive integers such that

$$\sum_{i=1}^{k} a_i = n.$$

For the sake of completeness we prove the following Lemma:

Lemma 5.7. *There are* $\binom{n-1}{k-1}$ *compositions of* n *into* k *parts.*

Proof. Let $(a_1, a_2, \dots, a_k)$ be composition of n into k parts. Define a mapping σ from the compositions of n into k parts to the $k-1$ element subsets of $[n-1]$ by

$$\sigma[(a_1, a_2, \dots, a_k)] = \{a_1, a_1 + a_2, \dots, a_1 + a_2 + \cdots + a_{k-1}\}.$$

It is clear that σ is a bijection. $\square$

We are now ready to prove the following Theorem first proved by T.V. Narayana [11], [12].

Theorem 5.8. *The number of Dyck paths in $\mathcal{C}_n$ with k peaks is the Narayana number $N(n, k-1)$, that is,*

$$\#\{w \in \mathcal{C}_n \mid \mathrm{P}(w) = k\} = N(n, k-1).$$

Proof. For all pairs $A = (a_1, a_2, \dots, a_k)$ and $B = (b_1, b_2, \dots, b_k)$ of compositions of $n+1$ and n into k parts, construct a sequence $s(A, B) = (a_1, -b_1, a_2, -b_2, \dots, a_k, -b_k)$. Consider $s(A, B)$ and $s(A', B')$ equivalent if $\pi_{2d}(s(A, B)) = s(A', B')$ for some permutation $\pi_{2d} \in \mathcal{S}_{2k}$. There are at most k members of the equivalence class $\overline{s}(A, B)$. If the order of π_{2d} is κ then we can partition $[2k]$ into classes $E_s = \{\pi_{2d}^i(s)\}_{i=0}^{\kappa-1}$. Now suppose that $\pi_{2d}(s(A, B)) = s(A, B)$ where $s(A, B) = (c_1, c_2, \dots c_{2k})$. Then we have

$$p, q \in E_s \Rightarrow c_p = c_q,$$

and consequently

$$\begin{aligned} \kappa \mid \sum_{\substack{E_s \\ 2 \nmid s}} c_s |E_s| &= \sum_{i=1}^{k} a_i = n+1 \quad \text{and} \\ \kappa \mid \sum_{\substack{E_s \\ 2 \mid s}} c_s |E_s| &= \sum_{i=1}^{k} b_i = n, \end{aligned}$$

where $x \mid y$ means x divides y. Since the greatest common divisor of n and $n+1$ is 1 we have that κ equals 1 and $\pi_{2d} = \iota$. We have shown that there are exactly k members of each equivalence class $\overline{s}(A, B)$. Since there are $\binom{n}{k-1}$ and $\binom{n-1}{k-1}$ compositions of $n+1$ and n respectively, there are

$$\frac{1}{k}\binom{n}{k-1}\binom{n-1}{k-1} = N(n, k-1)$$

equivalence classes. According to Lemma 5.5 there is exactly one member of each equivalence class such that for all $j \in [k]$, $\sum_{i=1}^{j}(a_i - b_i) \geq 1$. If we let w be the Dyck path beginning with $(a_1 - 1)$ u's followed by b_1 d's followed by a_2 u's and so on we have a bijection between the Dyck paths of length n with k peaks and the equivalence classes $\overline{s}(A, B)$ and the Theorem is proved. □

Let $w \in \mathcal{C}_n$. Since each non-initial u is either immediately preceded by a u or a d, i.e is the second letter that constitutes a valley or a double descent, we have $\mathrm{V}(w) + \mathrm{DA}(w) = n - 1$. Since a valley is uniquelly determined by two consequtive peaks we have $\mathrm{V}(w) = \mathrm{P}(w) - 1$. Since $N(n,k) = N(n, n-k-1)$ we have proved the following relationship:

$$\#\{w \in \mathcal{C}_n \mid \mathrm{V}(w) = k\} = \#\{w \in \mathcal{C}_n \mid \mathrm{DA}(w) = k\} = N(n,k).$$

A non-empty Dyck path w can uniquely be expressed as $w = uw_1dw_2$ where $w_1 \in \mathcal{C}_{k-1}$ and $w_2 \in \mathcal{C}_{|w|-k}$ and $2k$ is the least index greater than zero in w such that the height is zero. We call this decomposition of w the *first return decomposition* of w. This decomposition is often expressed as a set identity,

$$\mathcal{C}_\infty = \{1\} \bigcup \{u\}\mathcal{C}_\infty\{d\}\mathcal{C}_\infty,$$

where AB is the *concatenation* of A and B. If A and B are sets of words on the same alphabet then

$$AB = \{a_1 \cdots a_n b_1 \cdots b_k \mid a_1 \cdots a_n \in A \text{ and } b_1 \cdots b_k \in B\}.$$

In order to show that the distribution of even ascents and the distribution of odd ascents is given by the Narayana numbers we define a class of bijections on $\mathcal{C}_\infty$.

Definition 5.9. Let $\mathfrak{F}(\mathcal{C}_\infty)$ be the set of all length preserving bijections $\xi : \mathcal{C}_\infty \to \mathcal{C}_\infty$. Define two operations on $\mathfrak{F}(\mathcal{C}_\infty)$ as follows: If $\xi, \eta \in \mathfrak{F}(\mathcal{C}_\infty)$ then for $w \in \mathcal{C}_\infty$

$$\xi \bullet \eta(w) = \begin{cases} 1 & \text{if } w = 1, \\ u\xi(w_1)d\eta(w_2) & \text{if } uw_1dw_2 \text{ is the first return decomposition of } w. \end{cases}$$

$$\xi * \eta(w) = \begin{cases} 1 & \text{if } w = 1, \\ u\xi(w_2)d\eta(w_1) & \text{if } uw_1dw_2 \text{ is the first return decomposition of } w. \end{cases}.$$

One easily sees that the following relationships for the inverses of the operations hold:

$$\begin{aligned} (\xi \bullet \eta)^{-1} &= \xi^{-1} \bullet \eta^{-1} \text{ and} \\ (\xi * \eta)^{-1} &= \eta^{-1} * \xi^{-1}. \end{aligned}$$

Lemma 5.10. *For each pair of bijections ξ and η in $\mathfrak{F}(\mathcal{C}_\infty)$ there are unique bijections $\Upsilon, \Lambda \in \mathfrak{F}(\mathcal{C}_\infty)$ satisfying the relations*

$$\begin{aligned} \Upsilon &= (\Upsilon \circ \xi) \bullet (\Upsilon \circ \eta) \\ \Lambda &= (\Lambda \circ \xi) * (\Lambda \circ \eta). \end{aligned}$$

Proof. The Lemma follows easily by induction over the length of Dyck paths. There is only one bijection on $\mathcal{C}_0$, and it satisfies the recursion. Assume that there are unique functions Υ and Λ satisfying the relations for all Dyck paths smaller than $n > 0$. Then for $w \in \mathcal{C}_n$ define $\Lambda_n : \mathcal{C}_n \to \mathcal{C}_n$ by $\Lambda_n(w) = (\Lambda \circ \xi) * (\Lambda \circ \eta)(w)$. Hence

$$\bar{\Lambda}_n = \begin{cases} \Lambda_n(w) & \text{if } |w| = n, \\ \Lambda(w) & \text{if } |w| \neq n. \end{cases}$$

satisfies the relation for all Dyck paths of length at most n, and $\bar{\Lambda}_n$ is of course the only such function. Hence by induction we may define a unique bijection Λ satisfying the relation for all Dyck paths. The argument obviously holds for the recursion involving Υ. □

The reason for studying these operations is that it enables us to find bijections defined recursively on $\mathcal{C}_\infty$, according to the following consequence of the definition of the operations.

Lemma 5.11. *Let α and β be statistics on $\mathcal{C}_\infty$, $\xi, \eta \in \mathfrak{F}(\mathcal{C}_\infty)$ and let G be a mapping $G : \mathbb{Z}^4 \to \mathbb{Z}$.*

(i). *If $\alpha(1) = \beta(1)$ and for all $w = uw_1dw_2$ α and β satisfy*

$$\begin{aligned} \alpha(w) &= G(\alpha(w_1), \alpha(w_2), |w_1|, |w_2|) \text{ and} \\ \beta(w) &= G(\beta \circ \xi(w_1), \beta \circ \eta(w_2), |w_1|, |w_2|), \end{aligned}$$

then $\alpha \circ \Upsilon = \beta$ for the unique bijection defined by $\Upsilon = (\Upsilon \circ \xi) \bullet (\Upsilon \circ \eta)$

(ii). *If $\alpha(1) = \beta(1)$ and for all $w = uw_1dw_2$ α and β satisfy*

$$\begin{aligned} \alpha(w) &= G(\alpha(w_1), \alpha(w_2), |w_1|, |w_2|) \text{ and} \\ \beta(w) &= G(\beta \circ \xi(w_2), \beta \circ \eta(w_1), |w_2|, |w_1|), \end{aligned}$$

then $\alpha \circ \Lambda = \beta$ for the unique bijection defined by $\Lambda = (\Lambda \circ \xi) * (\Lambda \circ \eta)$.

Proof. Again the proof is by induction over the length of w. Assume that $\alpha \circ \Lambda(v) = \beta(v)$ for all Dyck paths v smaller than w, then

$$\begin{aligned} \alpha \circ \Lambda(w) &= \alpha(u\Lambda \circ \xi(w_2) d\Lambda \circ \eta(w_1)) = \\ &= G(\alpha \circ \Lambda \circ \xi(w_2), \alpha \circ \Lambda \circ \eta(w_1), |w_1|, |w_2|) = \\ &= G(\beta \circ \xi(w_2), \beta \circ \eta(w_1), |w_2|, |w_1|) = \beta(w). \end{aligned}$$

The first assertion is proved similarly. □

Definition 5.12. Let Γ and Ψ be the unique bijections in $\mathfrak{F}(\mathcal{C}_\infty)$ defined by

$$\begin{aligned} \Gamma &= \Gamma * \Gamma \text{ and} \\ \Psi &= (I \circ \Psi) \bullet \Psi, \end{aligned}$$

where $I = \iota * \iota$ and ι is the identity function.

The bijection Γ is actually an involution, which follows immediately from the relations for the inverses of the operations,

$$\Gamma^{-1} = (\Gamma * \Gamma)^{-1} = \Gamma^{-1} * \Gamma^{-1}$$

and since this relation uniquely defines Γ^{-1}, we have $\Gamma^{-1} = \Gamma$.

Proposition 5.13. *The number of even ascents and the number of odd ascents have a distribution given by the Narayana numbers. Moreover, the following identities hold:*

$$\begin{aligned} \mathrm{EA} \circ \Psi &= \mathrm{DA}, \\ \mathrm{V} \circ \Gamma &= \mathrm{DA}, \\ \mathrm{OA} \circ \Psi &= \mathrm{P}\,. \end{aligned}$$

Proof. The identities obviously hold for the empty word, hence let w be nonempty and $w = uw_1dw_2$ the first return decomposition of w. We have the following identities:

$$\begin{aligned} \mathrm{V}(w) &= \mathrm{V}(w_1) + \mathrm{V}(w_2) + \delta(w_2), \\ \mathrm{DA}(w) &= \mathrm{DA}(w_1) + \mathrm{DA}(w_2) + \delta(w_1), \\ \mathrm{EA}(w) &= \mathrm{OA}(w_1) + \mathrm{EA}(w_2) = |w_1| - \mathrm{EA}(w_1) + \mathrm{EA}(w_2), \end{aligned}$$

where $\delta(v)$ is zero if v is the empty word and one otherwise. That $\mathrm{V} \circ \Gamma = \mathrm{DA}$ now follows immediately from Lemma 5.11 if we let $G(x_1, x_2, x_3, x_4) = x_1 + x_2 + (1 - \delta_{x_4,0})$, where $\delta_{i,j}$ is the Kronecker delta. For a nonempty Dyck path w we have

$$\begin{aligned} \mathrm{EA}(w) + \mathrm{EA} \circ I(w) &= \mathrm{OA}(w_1) + \mathrm{EA}(w_2) + \mathrm{OA}(w_2) + \mathrm{EA}(w_1) = \\ &= |w_1| + |w_2| = |w| - 1. \end{aligned}$$

We thus have

$$\begin{aligned} \mathrm{EA}(w) &= \begin{cases} \mathrm{EA} \circ I(w_1) + \mathrm{EA}(w_2) & \text{if } w_1 = 1, \\ \mathrm{EA} \circ I(w_1) + \mathrm{EA}(w_2) + 1 & \text{otherwise}\,. \end{cases} = \\ &= \mathrm{EA} \circ I(w_1) + \mathrm{EA}(w_2) + \delta(w_1). \end{aligned}$$

Let $G(x_1, x_2, x_3, x_4) = x_1 + x_2 + (1 - \delta_{x_3,0})$ and apply Lemma 5.11. We thus have $DA \circ \Psi^{-1} = \mathrm{EA}$ for $\Psi^{-1} = (\Psi^{-1} \circ I) \bullet \Psi^{-1}$, which by the relationship for inverses gives the wanted bijection. Since

$$\begin{aligned} \mathrm{EA}(w) + \mathrm{OA}(w) &= |w| \\ \text{and} \quad \mathrm{DA}(w) + \mathrm{P}(w) &= \mathrm{DA}(w) + \mathrm{V}(w) + 1 = |w|, \end{aligned}$$

we have $\mathrm{OA} \circ \Psi = \mathrm{P}$. $\square$

The involution Γ can be used to prove yet another identity.

Definition 5.14. The number of *returns* of a Dyck path $w \in \mathcal{C}_n$ is defined to be the number of $i \in [2n]$ such that the height $h_i(w) = 0$. We define the statistic $\mathrm{NR}(w)$ to be the number of returns of w and the statistic $\mathrm{HFP}(w)$ to be the *height of the first peak* in w.

Proposition 5.15. *The statistics height of first peak and number of returns are equidistributed. Moreover*

$$\begin{aligned} \mathrm{HFP} \circ \Gamma &= \mathrm{NR}, \\ \#\{w \in \mathcal{C}_n \mid \mathrm{HFP}(w) = k\} &= \frac{k}{2n-k}\binom{2n-k}{n}. \end{aligned}$$

Proof. The following identities are immediate from the definition of the statistics. For $w = uw_1dw_2$,

$$\begin{aligned} \mathrm{HFP}(w) &= \mathrm{HFP}(w_1) + 1, \\ \mathrm{NR}(w) &= \mathrm{NR}(w_2) + 1. \end{aligned}$$

Applying Lemma 5.11 to the function $G = x_1 + 1$ gives the first statement of the Proposition. The number of Dyck paths of length n with its first peak at height k is equal to the number of positive paths from $(k+1, k-1)$ to $(2n, 0)$ which by a change of axis is the number of positive paths from $(0,0)$ to $(2n-k-1, k-1)$ which equals the ballot number

$$b(n-1, n-k) = \frac{k}{2n-k}\binom{2n-k}{n},$$

and the Proposition follows. □

Example 5.16. Let $w = uududduuddud$. Then

$$\begin{aligned} \Gamma(w) &= u\Gamma(uuddud)d\Gamma(udud) = \\ &= uu\Gamma(ud)d\Gamma(ud)du\Gamma(ud)d = \\ &= uuudduudduudd, \end{aligned}$$

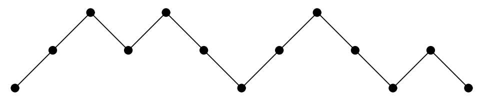
and

$$\begin{aligned} \Psi(w) &= uI \circ \Psi(udud)d\Psi(uuddud) = \\ &= uuuddduuddud, \end{aligned}$$

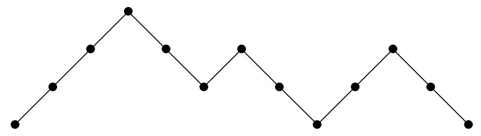
since

$$\begin{aligned} \Psi(u1dud) &= udud, \\ I(udud) &= uudd \text{ and} \\ \Psi(uuddud) &= uI \circ \Psi(ud)d\Psi(ud) = uuddud. \end{aligned}$$

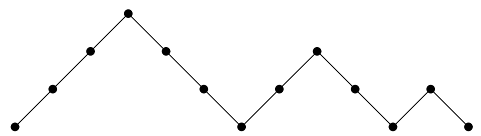
The Dyck paths are shown below:



$w = ududduuddud$



$\Gamma(w) = uuudduddduudd$



$\Psi(w) = uuudddduuddud$

6 The Catalan poset

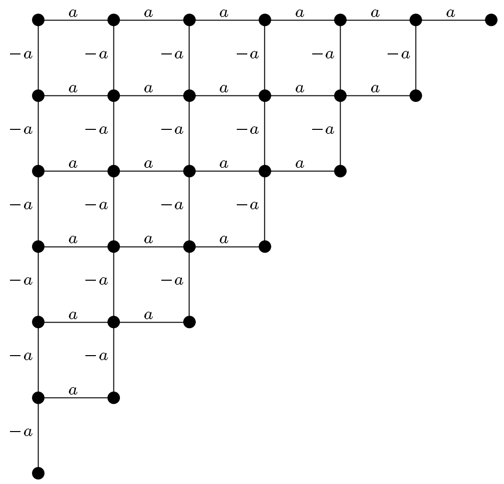
We now turn our attention to a class of posets closely connected to the Dyck paths. Let

$$\mathfrak{C}_n = \{(x, y) \in \mathbb{Z}^2 \mid 0 \leq x \leq y \leq n\}$$

$$\text{ordered by } (x, y) \leq (x', y') \text{ iff } x \leq x' \text{ and } y \leq y'.$$

be the *Catalan poset* of length $2n$. The connection between Dyck paths and $\mathfrak{C}_n$ is seen if we denote an edge between (x, y) and $(x, y+1)$ with a u and an edge between (x, y) and $(x+1, y)$ with a d. Then a maximal chain in $\mathfrak{C}_n$ is uniquely determined by its sequence of consequetive u's and d's. Moreover, the words that arise in this manner are exactly the Dyck paths of length n. The number of maximal chains in this poset is thus the Catalan number C_n, and a non-trivial fact is that the kth coordinate of the h-vector of $\Delta(\mathfrak{C}_n)$ is the Narayana number $N(n, k)$. We will show this. Let λ be the L-labeling of $\mathfrak{C}_n$ defined by

$$\begin{aligned}
&\lambda : C(\mathfrak{C}_n) \to \{-a, a\} \text{ where } -a < a \text{ and}\\
&\lambda[(x, y), (x, y+1)] = -a \text{ and}\\
&\lambda[(x, y), (x+1, y)] = a.
\end{aligned}$$



The unique rising unrefinable chain in $[(x_1, y_1), (x_2, y_2)]$ is of course

$$(x_1, y_1) \prec (x_1, y_1 + 1) \prec \cdots \prec (x_1, y_2) \prec (x_1 + 1, y_2) \prec \cdots \prec (x_2, y_2).$$

This is unique since if we choose to move horizontally before we have reached (x_1, y_2) we will have to move from a positive to a negative label. It is an L-labeling since moving horizontally implies a positive first label and moving vertically a negative first label. If we let $a = d$ and $-a = u$ then for a maximal chain $\mathbf{d}$, $\pi(\mathbf{d})$ is a unique Dyck path. A descent, $a, -a$, in $\pi(\mathbf{d})$ is a valley, du, in this path. The following Proposition now follows immediately from Corollary 4.7.

Proposition 6.1. *The kth coordinate of the h-vector of $\Delta(\mathfrak{C}_n)$ is*

$$h_k(\Delta(\mathfrak{C}_n)) = N(n, k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}$$

A shelling of $\Delta(\mathfrak{C}_n)$ can be thought of as a linear order Λ on $\mathcal{C}_n$. Recall that the cardinality of the restriction $r_\Lambda(w)$ of a facet w is equal to the number of facets in

$$2^w \cap \bigcup_{v <_\Lambda w} 2^v.$$

According to Proposition 2.6 and Proposition 6.1 the statistic $w \mapsto |r_\Lambda(w)|$ has the Narayana distribution. It is easy to see that for $w, v, v' \in \mathcal{C}_n$ we have

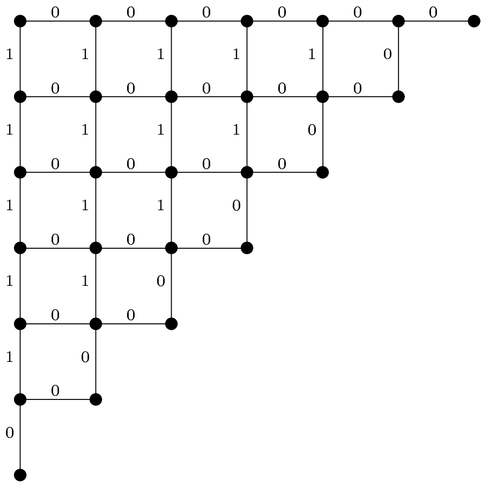
$$v \cap w = v' \cap w \text{ and } |v \cap w| = |v' \cap w| = |w| - 1 \Rightarrow v = v',$$

where $|w|$ denotes the cardinality of w when thought of as a chain in $\mathfrak{C}_n$. Hence $|r_\Lambda|$ satisfies:

$$|r_\Lambda(w)| = |\{v \in \mathcal{C}_n : v <_\Lambda w, \quad |w \cap v| = |w| - 1\}|. \tag{9}$$

One may wonder if there are other statistics apart from the number of valleys that correspond to shellings of the Catalan poset. We will now prove that the number of high peaks has the Narayana distribution by studying another L-labeling of $\Delta(\mathfrak{C}_n)$. That HP has the Narayanadistribution was first proved by E. Deutsch [7]. Let μ be the L-labeling of $\mathfrak{C}_n$ defined by

$$\begin{aligned}
&\mu : C(\mathfrak{C}_n) \to \{0,1\} \text{ where } 0 < 1 \text{ and} \\
&\mu[(x,y),(x+1,y)] = 0, \\
&\mu[(x,x),(x,x+1)] = 0 \text{ and} \\
&\mu[(x,y),(x,y+1)] = 1 \text{ if } x \neq y.
\end{aligned}$$



This labeling corresponds to a shelling where the first maximal chain is the diagonal and where we continuously move upwards to get the successive maximal chains. It is essentially the reverse order of the previous shelling. To find the unique rising for the interval $[(x_1,y_1),(x_2,y_2)]$ we distinguish between two situations. We move horizontally until we reach the diagonal or the point (x_2,y_1). If we reach the diagonal first we continue to move along the diagonal until we reach (x_2,y_1) and then move vertically until we reach (x_2,y_2). If we reach (x_2,y_1) first we turn and move vertically until we reach (x_2,y_2).

(i). The rising in the first case is

$$(x_1,y_1) \prec (x_1+1,y_1) \prec \cdots \prec (y_1-1,y_1) \prec \cdots \prec$$

$$\prec (x_2,x_2) \prec (x_2,x_2+1) \prec \cdots \prec (x_2,y_2).$$

(ii). The rising in the second case is

$$(x_1,y_1) \prec (x_1+1,y_1) \prec \cdots \prec (x_2,y_1) \prec (x_2,y_1+1) \prec \cdots \prec (x_2,y_2).$$

This rising is unique, since if we choose to move vertically before the given point we have to move from an edge labelled 1 to an edge labelled 0. The labeling is an L-labeling since the horizontal edges are labelled zero and if the unique rising is one in which we start by moving vertically, then the end point of the interval must be at the same x-coordinate as the starting point, hence there is only one chain in the interval. The descents, $1, 0$, in the Jordan Hölder sequence $\pi(\mathbf{c})$ correspond to the high peaks in the corresponding Dyck path.

There is yet another statistic with the Narayana distribution connected to a shelling of $\Delta(\mathfrak{C}_n)$. This is not a lexicographical shelling but a shelling induced by a certain partial order on $\mathcal{C}_\infty$, or rather on the maximal chains of $\cup_{n\geq 0}\mathfrak{C}_n$, but since these structures are almost identical we will stick to the notation of the former (the notation involving Dyck paths).

Definition 6.2. Let $\mathcal{A}(\mathcal{C}_\infty)$ be the following set of mappings on $\mathcal{C}_\infty$:

$$\mathcal{A}(\mathcal{C}_\infty) = \bigcup_{i\geq 1}\{\theta_i, \phi_i\},$$

where θ_i and ϕ_i are defined for $w = a_1 \cdots a_{2n}$ by

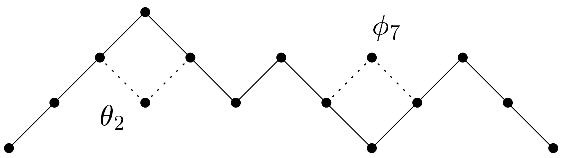
$$\theta_i(w) = \begin{cases} a_1 \cdots a_{i-1} udu a_{i+3} \cdots a_{2n} & \text{if } a_i a_{i+1} a_{i+2} = uud, \\ w & \text{otherwise ,} \end{cases}$$

$$\phi_i(w) = \begin{cases} a_1 \cdots a_{i-1} dud a_{i+3} \cdots a_{2n} & \text{if } a_i a_{i+1} a_{i+2} = ddu, \\ w & \text{otherwise ,} \end{cases}$$

and let $\mathcal{A}^*(\mathcal{C}_\infty)$ be the set of all compositions of mappings in $\mathcal{A}(\mathcal{C}_\infty)$;

$$\mathcal{A}^*(\mathcal{C}_\infty) = \{\rho : \mathcal{C}_\infty \to \mathcal{C}_\infty \mid \rho = \rho_k \circ \rho_{k-1} \circ \cdots \circ \rho_1$$
$$\text{where } \rho_i \in \mathcal{A}(\mathcal{C}_\infty) \text{ for all } i \in [k]\}.$$

The mappings thus change certain valleys to peaks and certain peaks to valleys. We may define a partial order Λ on $\mathcal{C}_\infty$ as follows.



Definition 6.3. For two Dyck paths v and w, let $v \leq_\Lambda w$ if and only if $\rho(w) = v$ for some mapping $\rho \in \mathcal{A}^*(\mathcal{C}_\infty)$.

In other words, we consider v to be smaller than or equal to w if v can be obtained from w by turning a number of valleys into peaks and peaks into valleys in w according to Definition 6.2.

Lemma 6.4. *The relation Λ above is a partial order on $\mathcal{C}_\infty$.*

Proof. The only non-trivial property of Λ is the anti-symmetry, that is:

$$v \leq_\Lambda w \text{ and } w \leq_\Lambda v \Rightarrow v = w.$$

It suffices to prove that for all w we have

$$\rho \in \mathcal{A}(\mathcal{C}_\infty), \zeta \in \mathcal{A}^*(\mathcal{C}_\infty) \text{ and } \rho(w) \neq w \Rightarrow \zeta \circ \rho(w) \neq w.$$

Since for any Dyck path the number of double ascents and the number of double descents are equal, it easily follows that for any $\rho \in \mathcal{A}(\mathcal{C}_\infty)$ the number of double ascents in $\rho(w)$ is less than or equal to the number of double ascents in w. Let $w = a_1 \cdots a_{2n}$. If $a_i \cdots a_{i+3} = uudd$ and $\rho = \theta_i$, or $a_i \cdots a_{i+3} = dduu$ and $\rho = \phi_i$ then

$$\mathrm{DA} \circ \rho(w) < \mathrm{DA}(w).$$

Let $\sigma(w)$ be the sum of the position of all double descents and double ascents in $w = a_1 \cdots a_{2n}$;

$$\sigma(w) = \sum_{i=1}^{2n} p(a_i a_{i+1}),$$

where $p(a_i a_{i+1}) = i$ if $a_i a_{i+1}$ is a double descent or a double ascent and 0 otherwise. If $a_i \cdots a_{i+3} = uudu$ and $\rho = \theta_i$, or if $a_i \cdots a_{i+3} = ddud$ and $\rho = \phi_i$ then

$$\sigma \circ \rho(w) = \sigma(w) + 2.$$

This suffices to prove the Lemma, since however we chose to apply the mappings we will alter an invariant (if $\rho(w) \neq w$). □

Remark 6.5. If two Dyck paths v and w intersect maximally then all coordinates are the same but one, where v has a valley and w has a high peak, or the other way around. It follows that w and v intersect maximally if and only if $w \neq \rho(w) = v$ or $v \neq \rho(v) = w$ for some $\rho \in \mathcal{A}(\mathcal{C}_\infty)$.

The only Dyck path in $\mathcal{C}_n$ without any LNFSs is the alternating path $udud \cdots ud$, hence the poset $(\mathcal{C}_n, \Lambda)$ has a least element. It is easily seen that for $n > 3$, $(\mathcal{C}_n, \Lambda)$ does not have a greatest element. We want to expand Λ into a linear order that induces a shelling of $\mathcal{C}_n$ (the maximal chains of $\Delta(\mathfrak{C}_n)$). If $n > 3$ this expansion is not unique.

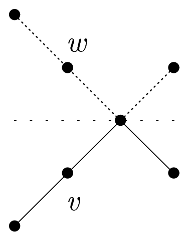
If $a_i a_{i+1} a_{i+2}$ is a long non-final sequence in $w = a_1 \cdots a_{2n}$ we say that a Dyck path v *touches* this LNFS if the height $h_{i+1}(v)$ equals the height $h_{i+1}(w)$.

Lemma 6.6. *Let $v = a_1 \cdots a_{2n}$ and $w = b_1 \cdots b_{2n}$ be Dyck paths of the same length. Then* (i). *If v touches all the LNFSs of w and w touches all the LNFSs of v then* $w = v$.

(ii). *If v touches all the LNFSs of w then* $w \leq_\Lambda v$.

Proof. The first statement is of course implied by the second, but we will use it to prove the second statement.

(i) A Dyck path is uniquely determined by the position and nature (*uud* or *ddu*) of its long non-final sequences. Thus it suffices to prove that if v and w are as in (i) then v can not have a subword $a_i a_{i+1} a_{i+2} = uud$ where w has a subword $b_i b_{i+1} b_{i+2} = ddu$. In order to get a contradiction suppose that we have the situation above (see figure below). Then $h_{i-1}(v) = h_{i+1}(v) - 2$ and $h_{i-1}(w) = h_{i+1}(w) + 2$. If we let k be the greatest integer smaller than i such that $h_k(v) = h_k(w)$ then it is clear that either v or w has a LNFS in the interval $[k, i-1]$ since otherwise one of them would not reach the required height at position $i-1$. This is a contradiction since v and w were supposed to satisfy (i).



(ii) If $v = w$ then we are done. Suppose that $v \neq w$. We want to prove that there is a mapping $\rho_1 \in \mathcal{A}(\mathcal{C}_\infty)$ such that $\rho_1(v) \neq v$ and $\rho_1(v)$ touches all the LNFSs of w. According to (i) there is at least one LNFS of v that w does not touch, hence let $\rho_1 \in \mathcal{A}(\mathcal{C}_\infty)$ be the mapping that changes this LNFS. If $\rho_1(v) = w$ we are done and if not we continue the procedure and get another mapping $\rho_2 \in \mathcal{A}(\mathcal{C}_\infty)$, such that $\rho_2 \circ \rho_1(v) <_\Lambda \rho_1(v) <_\Lambda v$. Since $\mathcal{C}_n$ is finite we can not have an infinite sequence of Dyck paths, hence we finally have for some $k > 0$:

$$v >_\Lambda \rho_1(v) >_\Lambda \rho_2 \circ \rho_1(v) >_\Lambda \cdots >_\Lambda \rho_k \circ \rho_{k-1} \circ \cdots \circ \rho_1(v) = w,$$

and the Lemma is proved. □

We are now ready to prove that LNFS has the Narayana distribution, which was first proved by G. Kreweras and P. Moszkowski [9] by induction over the length of the Dyck paths, and later bijectively by S. Benchekroun and P. Moszkowski [4]. As announced we prove it by studying a particular shelling.

Proposition 6.7. *For all $n \in \mathbb{N}$ there is a shelling $w_1, w_2, \ldots, w_{C_n}$ of $\Delta(\mathfrak{C}_n)$ such that*

$$|r(w_i)| = \mathrm{LNFS}(w_i) \quad \text{(see Definition 2.3)}.$$

In particular the statistic LNFS *has the Narayana distribution.*

Proof. Let Ω be any linear order on the maximal chains of $\Delta(\mathfrak{C}_n)$ such that for all $v, w \in \mathcal{C}_n$

$$v <_\Lambda w \Rightarrow v <_\Omega w.$$

We want to show that Ω induces a shelling, that is,

$$v <_\Omega w \Rightarrow v \cap w \subseteq w' \cap w = w \setminus \{x\} \text{ for some } w' <_\Omega w \text{ and } x \in \mathfrak{C}_n$$

Suppose $v <_\Omega w$. According to Lemma 6.6 there is a long non-final sequence in w which v does not touch. Hence we obtain a preceding facet intersecting w maximally if we apply a suitable function $\rho \in \mathcal{A}(\mathcal{C}_\infty)$ to w, namely the mapping that changes the LNFS in question. Thus we have

$$w \cap v \subseteq \rho(w) \cap w = w \setminus \{x\}.$$

Hence Ω induces a shelling, and it follows from Remark 6.5 that the number of preceding facets intersecting w maximally is the number of LNFSs in w. According to (9) we now have:

$$|r_\Omega(w)| = \mathrm{LNFS}(w),$$

where $r_\Omega(w)$ is the restriction of w, which proves the Proposition. ☐

7 Some generating functions

Generating functions are powerful tools in combinatorics and some of the generating functions connected to positive paths are rather easy to find and often have a pleasant appearance. The generating function for the Narayana numbers is of course defined to be

$$N(x,t) = \sum_{n,k} N(n,k) x^k t^n = \sum_{w \in \mathcal{C}_\infty} x^{\mathrm{V}(w)} t^{|w|}$$

where $\mathcal{C}_\infty = \cup_{n \geq 0} \mathcal{C}_n$. If $w = uw_1dw_2$ is the first return decomposition of w then

$$\begin{aligned} |w| &= |w_1| + |w_2| + 1, \\ \mathrm{V}(w) &= \mathrm{V}(w_1) + \mathrm{V}(w_2) + \delta(w_2). \end{aligned}$$

Hence if $\mathrm{wt}(w; x, t) = x^{\mathrm{V}(w)} t^{|w|}$ we have

$$\mathrm{wt}(w; x, t) = x^{\delta(w_2)} t \, \mathrm{wt}(w_1; x, t) \, \mathrm{wt}(w_2; x, t)$$

We can now find a relation for the generating function:

$$\begin{aligned}
N(x,t) &= \sum_{w\in\mathcal{C}_\infty} \mathrm{wt}(w;x,t) = 1 + t\sum_{w_1,w_2\in\mathcal{C}_\infty} x^{\delta(w_2)}\,\mathrm{wt}(w_1;x,t)\,\mathrm{wt}(w_2;x,t)\\
&= 1 + xt\sum_{w_1,w_2\in\mathcal{C}_\infty} \mathrm{wt}(w_1;x,t)\,\mathrm{wt}(w_2;x,t) -\\
&- xt\sum_{w_1\in\mathcal{C}_\infty}\mathrm{wt}(w_1;x,t) + t\sum_{w_1\in\mathcal{C}_\infty}\mathrm{wt}(w_1;x,t) =\\
&= 1 + xtN^2(x,t) - xtN(x,t) + tN(x,t).
\end{aligned}$$

It should be pointed out that the equidistribution of the statistics defined in the previous sections can be proved in a generating function setting. We just have to show that the generating functions of the different statistics satisfy the recursion above, since it uniquely determines $N(x,t)$. Using the Lagrange inversion formula one can actually obtain the closed formula for $N(n,k)$ from this recursion. For a generating function approach to these problems see D. Zeilberger's article [14]. The recursion formula above enables us to express the generating function for the Narayana numbers as the following formal continued fraction:

$$N(x,t) = \cfrac{1}{1-\cfrac{t}{1-xtN(x,t)}} = \cfrac{1}{1-\cfrac{t}{1-\cfrac{xt}{1-\cfrac{t}{1-\cfrac{xt}{\ddots}}}}} \tag{10}$$

Let $C(t) = \sum_{n\geq 0} C_n t^n$ be the generating function for the Catalan numbers. It is clear that $C(t) = N(1,t)$ and we get the relationship

$$C(t) = 1 + tC^2(t).$$

If we let $x = 1$ in (10) we see that the generating function for the Catalan numbers can be written as the formal continued fraction

$$C(t) = \cfrac{1}{1-\cfrac{t}{1-\cfrac{t}{1-\cfrac{t}{1-\cfrac{t}{\ddots}}}}}$$

Let $\mathcal{B}$ be the set of all *ballot paths*, that is, the set of all positive paths that start at the origin. The ballot paths may be expressed as a union of disjoint sets as follows:

$$\begin{aligned}\mathcal{B} &= \{1\}\bigcup\{u\}\mathcal{C}_\infty\{d\}\mathcal{B}\bigcup\{u\}\mathcal{B},\\ \mathcal{B} &= \mathcal{C}_\infty\bigcup\mathcal{C}_\infty\{u\}\mathcal{B}.\end{aligned}$$

The first decomposition arises from the fact that a ballot path either touches or does not touch the x-axis. The second decomposition says that a ballot path is or is not a Dyck path. Let $\mu(w;x,t) = x^{|w|_d}t^{|w|_u}$. From the first decomposition we get that generating function for the ballot numbers satisfies

$$\begin{aligned}B(x,t) &= \sum_{n,k} b(n,k)t^n x^k = \sum_{w\in\mathcal{B}}\mu(w;x,t) =\\ &= 1+\sum_{w_1\in\mathcal{C}_\infty}\sum_{w_2\in\mathcal{B}} xt(xt)^{|w_1|}\mu(w_2;x,t)+\sum_{w\in\mathcal{B}} t\mu(w;x,t) =\\ &= 1+xtC(xt)B(x,t)+tB(x,t).\end{aligned}$$

Thus we may write $B(x,t)$ as the following continued fraction:

$$B(x,t) = \cfrac{1}{1-t-\cfrac{xt}{1-\cfrac{xt}{1-\cfrac{xt}{1-\cfrac{xt}{\ddots}}}}}$$

From the second decomposition we similarly get

$$B(x,t) = C(xt)+tC(xt)B(x,t),$$

and thus,

$$B(x,t) = \frac{C(xt)}{1-tC(xt)}.$$

8 Patterns and Dyck paths

The Dyck paths in $\mathcal{C}_\infty$ can be associated with a certain kind of permutations. Let $\mathcal{S}_\infty = \cup_{n\geq 0}\mathcal{S}_n$, where the empty permutation, ε, is considered a permuation of length 0. For a permutation π on $S\subset\mathbb{N}$ let $\overline{\pi}$ be the corresponding permutation in $\mathcal{S}_n$, where $n=|S|$, such that the order relations among the elements are preserved. Thus $\overline{5372} = 3241$.

Definition 8.1. A *pattern* is any permutation τ, and we say that a permutation $\pi = a_1a_2\cdots a_n$ has k *occurrences* of τ ($|\tau| = t$) if there are exactly k subwords $w = a_{i_1}a_{i_2}\cdots a_{i_t}$ of π such that $i_1 < i_2 < \cdots < i_t$ and $\overline{w} = \tau$. In symbols this is expressed by $(\tau)\pi = k$. If $(\tau)\pi = 0$ we say that π *avoids* τ.

Let $\mathcal{S}_\infty(132) = \{\pi \in \mathcal{S}_\infty \mid (132)\pi = 0\}$ and $\mathcal{S}_n(132) = \{\pi \in \mathcal{S}_\infty(132) \mid |\pi| = n\}$. Since every nonempty permutation $\pi \in \mathcal{S}_n(132)$ is to avoid the pattern 132 it may be expressed as $\pi = \pi_1 n \pi_2$ where $\overline{\pi_1}, \overline{\pi_2} \in \mathcal{S}_\infty(132)$ and each element in π_1 is greater than all elements in π_2. On the other hand, if we have such a permutation it is clear that it avoids 132. We call $\pi_1 n \pi_2$ the *first return decomposition* of π. We know that every non-empty Catalan-path w can be uniquely written as uw_1dw_2. This enables us to define a bijection.

Proposition 8.2. *We may define a bijection $\Phi : \mathcal{S}_\infty(132) \to \mathcal{C}_\infty$ recursively as follows*

$$\Phi(\varepsilon) = 1 \text{ and } \Phi(\pi) = u\Phi(\overline{\pi_1})d\Phi(\overline{\pi_2})$$

Moreover, for all π we have $|\pi| = |\Phi(\pi)|$ and $|\mathcal{S}_n(132)| = C_n$, the Catalan number.

Proof. We need to prove that the restriction of Φ to $\mathcal{S}_n(132)$ is a bijection with the given properties for all $n \in \mathbb{N}$. Since Φ is defined in a recursive manner this is trivially done by induction over n. $\square$

This bijection is the same as Ψ described in [8], although Krattenthaler's definition is not recursively given.

Example 8.3. Let $\pi = 546231 \in \mathcal{S}_\infty(132)$ then

$$\Phi(\pi) = u\Phi(21)d\Phi(231) = uud\Phi(1)du\Phi(1)d\Phi(1) = uududduuddud.$$

The Dyck path $\Phi(\pi)$ is thus the same as w in Example 5.16.

Definition 8.4. Let $\pi = a_1a_2\cdots a_n$ be a permutation in $\mathcal{S}_n$. An $i \in [n-1]$ is a *descent* in π if $a_i > a_{i+1}$ and an *ascent* if $a_i < a_{i+1}$. We define the statistic $(ba)\pi$ to be the number of descents in π and $(ab)\pi$ to be the number of ascents in π.

The results concerning the equidistribution of statistics defined on Dyck paths are of course directly translated to concern statistics on $\mathcal{S}_\infty(132)$. The following Corollary is an immediate consequence of Lemma 5.11.

Corollary 8.5. *For each $\pi \in \mathcal{S}_\infty(132)$ the number of valleys in $\Phi(\pi)$ equals the number of descents in π and the number of double ascents in $\Phi(\pi)$ equals the number of ascents in π. In particular, the distribution of ascents and descents in $\mathcal{S}_n(132)$ is given by the Narayana numbers, that is,*

$$\begin{aligned}&\#\{\pi \in \mathcal{S}_n(132) \mid (ab)\pi = k\} = \\ &\#\{\pi \in \mathcal{S}_n(132) \mid (ba)\pi = k\} = N(n,k).\end{aligned}$$

Proof. Since for all permutations π and Dyck paths w we have

$$\begin{aligned} \mathrm{V}(w) + \mathrm{DA}(w) &= |w| - 1 \text{ and} \\ (ab)\pi + (ba)\pi &= |\pi| - 1, \end{aligned}$$

it will suffice to prove the statement for the ascents. For each $\pi \neq \varepsilon$, with $|\pi| = n$, let $\pi = \pi_1 n \pi_2$ be the first return decomposition of π, then

$$(ab)\pi = (ab)\overline{\pi_1} + (ab)\overline{\pi_2} + \delta(\overline{\pi_1}),$$

where $\delta(\pi) = 0$ if π is the empty word and is equal to one otherwise. On the other hand if $w = uw_1dw_2$ is the first return decomposition of a non-empty Catalan sequence then

$$\mathrm{DA}(w) = \mathrm{DA}(w_1) + \mathrm{DA}(w_2) + \delta(w_1).$$

The Corollary now follows from Lemma 5.11 since $\mathrm{DA} \circ \Phi(\varepsilon) = ab(\varepsilon) = 0$ and

$$\begin{aligned} \mathrm{DA} \circ \Phi(\pi) &= \mathrm{DA}(u\Phi(\overline{\pi_1})d\Phi(\overline{\pi_2})) = \\ &= \mathrm{DA} \circ \Phi(\overline{\pi_1}) + \mathrm{DA} \circ \Phi(\overline{\pi_2}) + \delta \circ \Phi(\overline{\pi_1}). \end{aligned}$$

□

A *right to left maximum* in a permutation π is an index i such that $\pi(i) > \pi(j)$ for all $j > i$. The number of right to left maxima in a permutation π is denoted $\mathrm{RLM}(\pi)$. If $\pi = \pi_1 n \pi_2$ is the first return decomposition of π then

$$\mathrm{RLM}(\pi) = \mathrm{RLM}(\overline{\pi_2}) + 1.$$

We thus have the following result for the distribution of RLM over $\mathcal{S}_\infty(132)$.

Corollary 8.6. *For all $\pi \in \mathcal{S}_\infty(132)$ we have*

$$\mathrm{NR} \circ \Phi(\pi) = \mathrm{RLM}(\pi).$$

In particular the distribution of RLM *over $\mathcal{S}_n(132)$ is*

$$\#\{\pi \in \mathcal{S}_n(132) \mid \mathrm{RLM}(\pi) = k\} = \frac{k}{2n-k}\binom{2n-k}{n}.$$

Proof. The Corollary follows from Proposition 8.2, Lemma 5.11 and Proposition 5.15. □

References

[1] D. André, *Solution directe du probléme résolu par M.J. Bertrand*, C.R. Acad. Sci. (Paris) **105**, 436-437 (1887).

[2] L. W. Beineke and Frank Harary, *The genus of the n-cube.* Canad. Math. J. **17** (1965) 494-496.

[3] L. W. Beineke and Frank Harary, *Inequalities involving the genus of a graph and its thicknesses.* Proc. Glasgow Math. Assoc. **7** (1965) 19-21.

[4] S. Benchekroun and P. Moszkowski, *A bijective proof of an enumerative property of legal bracketings.* Discrete Math. **176** (1997) 273-277.

[5] J. Bertrand, *Solution d'un probléme.* C.R. Acad. Sci. (Paris) **105**, 369 (1887).

[6] A. Björner, *Shellable and Cohen-Macauley partially ordered sets.* Trans. Amer. Math. Soc. **260** (1980) 159-183.

[7] E. Deutsch, *A bijection on Dyck paths and its consequence*, Discrete Math. **179** (1998) 253-256.

[8] C. Krattenthaler, *Permutations with restricted patterns and Dyck paths*, arXiv:math.CO/0002200 24 Feb 2000.

[9] G. Kreweras and P. Moszkowski, *A new enumerative property of the Narayana numbers.* J. Statist. Plann. Inference **14** (1986) 63-67.

[10] T. V. Narayana, *Sur les treilles formes par les partitions d'un entier*, C.R. Acad. Sci. (Paris) **240-1** (1955), 1188-9.

[11] T. V. Narayana, *A partial order and its application to probability theory*, Sankhya, **21** (1959) 91-98.

[12] T. V. Narayana, *Lattice path combinatorics with statistical applications*, Univ. of Toronto Press, 1979.

[13] R. Stanley, *Combinatorics and Commutative Algebra*, 2nd ed., Birkhäuser. Boston 1996.

[14] D. Zeilberger, *Six etudes in generating functions.* Intern. J. Computer Math. **29** 201-215 (1989).