Combinatorics of a Polynomial Riemann Hypothesis

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Outline

- T-transform and Polynomial Riemann Hypotheses
- 3-term and 4-term recursions, interleaving, conjectures
- The asymptotic recursion
- Moments and Hankel determinants
- Alternating Sign Matrices
- World's longest certificate
- Hankel paths and ASM with vertical symmetry
- More results and conjectures

The T-transform

Let $g(x) \in \mathbb{R}[x]$. Define $T[g](s)$ linearly on the power basis by

$$\begin{aligned} T[1](s) &= 1 \\ T[x^n](s) &= s(s+1)\cdots(s+n-1)/n! \end{aligned}$$

- In terms of the complex integral transform
$$T[g](s)\frac{\pi}{\sin(\pi s)} = \int_0^1 x^s(1-x)^{1-s}g(x)\frac{dx}{x(1-x)}.$$
- If $g(x) = g(1-x)$ then $T[g](s) = T[g](1-s)$.
- Especially interesting: The cases for which $T[g](s)$ satisfies a Riemann hypothesis; i.e., the zeros $\rho = \beta + i\gamma$, satisfy $\beta = \frac{1}{2}$.
- Example
$$L_n(s) = \sum_{k=0}^{n} (-\tfrac{1}{2})^{n-k}\binom{n}{k} \cdot \frac{(s)_k}{k!} \;=\; T[(x-\tfrac{1}{2})^n](s).$$

Elementary observations about the T-transform

- $Tp(x) \in Rh \Rightarrow p(x) = x^n - \frac{n}{2}x^{n-1} + \cdots$
- From the integral transform characterization

$$T(1-x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^{n-k} \cdot \frac{(x)_k}{k!} = \frac{(1-x)_n}{n!}$$

- Multiplicative (non)-properties
  - $Tp_1, Tp_2 \in Rh \not\Rightarrow T(p_1 p_2) \in Rh$
  - $Tp^n(s) \in Rh \not\Rightarrow Tp(s) \in Rh$
- $Tp(s) \in Rh \Rightarrow p(x) = q(x) + q(1-x)$

A general result: Redmond gave an analytic proof of the following:

Theorem: *If $g(x) \in \mathbb{R}[x]$ satisfies a Riemann hypothesis, then so does its T-transform $T[g](s)$.*

Linear Case

In particular,

$$T[(x+r)^n + (1-x+r)^n](s)$$

are in Rh, $\forall n > 1$ and $\forall r \in \mathbb{R}$.

A substantial amount of numerical evidence indicates that a great deal more is true.

Two important phenomena are involved:

- positivity
- interlacing

These are inaccessible by analytic methods,

- but -

alternate (combinatorial) approaches lead to a number of surprising connections.

The approach: 3-term recursions

Let $q_n(w) = T[(x+r)^n](w+\frac{1}{2})$
$T[(x+r)^n + (1-x+r)^n](s) \in Rh \Leftrightarrow$

$$q_n(w) + q_n(-w)$$

has purely imaginary zeros. The proof idea:

- $r \to -(1+r) \;\Rightarrow$ it is enough to consider $r \geq -\frac{1}{2}$.
- Theorem: *A normalized version $p_n(w)$ of $q_n(w)$ satisfy*

$$p_n(w) = (w - \alpha_n) \cdot p_{n-1}(w) - \beta_n \cdot p_{n-2}(w),$$

with $p_0(w) = 1,\; p_1(w) = w + r + \frac{1}{2}$,

$$\begin{aligned} \alpha_n &= (1-2n)(r+\tfrac{1}{2}) \\ \beta_n &= (n-1)^2 r(r+1) \end{aligned}$$

- For $r > 0$, $\alpha_n \in \mathbb{R}$ and $\beta_n > 0 \;\Rightarrow p_n(w)$ has real, simple roots (Favard's theorem).

- Use the recurrence to show that if we write
$$q_n(w) = \frac{1}{n!2^n} \sum_{i,j \geq 0} c_{ij} w^i r^j$$
then the $c_{i,j}$ are *nonnegative* integers. Since the roots of $q_n(w)$ are real, they must be negative.

- Lemma: *If the real parts of the zeros of $Q_n(w) \in \mathbb{R}[w]$ are negative, then the zeros of $Q_n(w) + Q_n(-w)$ are purely imaginary.*

- For $-\frac{1}{2} < r < 0$, the roots of $q_n(w)$ have negative *real parts.*

- For $r = -\frac{1}{2}$, we get the polynomials $L_n(s)$.

Quadratic Case

Comes from the family $P_n(s,r) = T[(x(x-1)+r)^n](s), \quad r \geq \frac{1}{4}$.

- $P_n(\frac{1}{2}+it, r)$ are real polynomials in t^2.

Put $u = -t^2$ and set $p_n(u,r) = P_n(\frac{1}{2}+it, r)$.

- Theorem: *The p_n satisfy a 4-term recursion.*

The Quadratic Polynomial Riemann hypothesis

For $r \geq \frac{1}{4}$, the u-zeros of $p_{n+1}(u,r)$ are negative and interlace the u-zeros of $p_n(u,r)$. Furhermore if

$$p_n(u, R+\tfrac{1}{4}) = \sum_{i,j \geq 0} c_{i,j} u^i R^j,$$

then the coefficients $c_{i,j}$ are nonnegative integers.

Review: 3-term recursions

$$q_n = (x - c_n)q_{n-1} - \lambda_n q_{n-2}, \quad q_{-1} = 0, q_0 = 1, \quad c_n, \lambda_n \in \mathbb{R}.$$

$\Rightarrow \exists$ a unique linear functional $\mathcal{L}$ on $\mathbb{R}[x]$ such that

$$\mathcal{L}[1] = \lambda_1, \quad \mathcal{L}[q_m q_n] = 0 \;\; m \neq n, \quad \mathcal{L}[q_n^2] = \lambda_1 \lambda_2 \cdots \lambda_{n+1}$$

- $\{q_n\}$ is an orthogonal wrt $\mathcal{L}$ if the $\lambda_n \neq 0$.
- $\mathcal{L} > 0$ (positive definite) iff all $\lambda_n > 0$.
- The zeros of the q_{n+1} are real and simple and interlace the zeros of q_n.
- If we specify the moments of $\mathcal{L}$ by $\mu_n = \mathcal{L}[x^n]$ (and take $\mu_0 = \lambda_1 = 1$), then $\mathcal{L} > 0$ iff the Hankel determinants

$$\Delta_n = \Delta_n[\mu_{i+j}]_{0 \le i,j \le n}$$

  are positive for $n = 0, 1, \ldots$.

The Asymptotic Recursion

Renormalized p_n-recursion of the Quadratic Case for large n.

Why? 4-term recursions $\Rightarrow$ orthogonality is lost $\Rightarrow$ no guarantee of real roots (let alone negative).

The asymptotic recursion is $\qquad q_n = xq_{n-1} - Cq_{n-2} - q_{n-3}$,
where $q_{-2} = q_{-1} = 0$, $q_0 = 1$, and

$$C = \frac{8r(6r-1)}{[16r^2(4r-1)]^{\frac{2}{3}}}.$$

In the quadratic Case, $r \geq \frac{1}{4}$. As r runs from $\frac{1}{4}$ to ∞, $C(r) \searrow 3$, and $C = 3$ is a critical value.

[3-Conjecture] *The q_n, $n = 1, 2, \ldots$, have real zeros iff $C \geq 3$. Moreover, in this case the zeros of q_{n+1} interlace the zeros of q_n.*

We *are* able to prove this conjecture in the case that $C \geq 3$.

The 6-conjecture

- The 3-Conjecture comes from T-transform of $(x(x-1)+r)^n$, $(r \geq \frac{1}{4})$, and the related normalized/asymptotic 4-term recursion
- Powers of other polynomials invariant under $x \to 1-x$ give higher order sequences
- What values of the parameters guarantee a Riemann hypothesis?

Example:
5-term recursion: $q_n = q_n(x, C)$ with $q_{-3} = q_{-2} = q_{-1} = 0,\ q_0 = 1$

$$q_n = xq_{n-1} - Cq_{n-2} + 4q_{n-3} - q_{n-4}, \qquad (n \geq 1)$$

[The 6-Conjecture] *The polynomials q_n have real zeros iff $C \geq 6$. In this case, the zeros of q_{n+1} interlace the zeros of q_n.*

Higher order recursions & conjectures

Numerical evidence $\Rightarrow$ other polynomial sequences depending on a single parameter C have real zeros if $C \geq$ *critical value*.

$$\begin{aligned} q_n &= x q_{n-1} - C q_{n-2} - q_{n-3} \\ q_n &= x q_{n-1} - C q_{n-2} + 4 q_{n-3} - q_{n-4} \end{aligned}$$

with $C \geq 3$, and $C \geq 6$, respectively.
Conjecture: *Critical coefficients that are at work come from binomial coefficients, e. g.*

$$\begin{aligned} 3, 1 &= \textit{for 4-term recursions,} \\ 6, 4, 1 &= \textit{for 5-term recursions.} \end{aligned}$$

[The $\binom{k}{k/2}$-Conjecture] For $(k+1)$-term recursions.

Moments

The moments $\mu_n = \mu_n(C) = \mathcal{L}_C[x^n]$ of the functional $\mathcal{L}$ associated to

$$q_n = xq_{n-1} - Cq_{n-2} - q_{n-3}$$

First few μ_n:

$$\begin{aligned}
\mu_0 &= 1 \\
\mu_1 &= 0 \\
\mu_2 &= C \\
\mu_3 &= 1 \\
\mu_4 &= 2C^2 \\
\mu_5 &= 5C \\
\mu_6 &= 3 + 5C^3
\end{aligned}$$

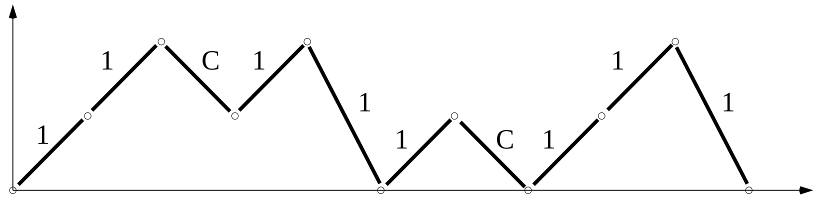
We give 9 different characterizations of the μ_n.

Theorem: *The moments μ_n of the functional $\mathcal{L}$ are given by any of the following expressions:*

1. *The sum of the weights (product) of all lattice paths from the origin to the point $(n,0)$ with elementary steps*

$$\begin{aligned} (a,b) &\to (a+1,b+1) && \text{with weight } 1, \\ (a,b) &\to (a+1,b-1) && \text{with weight } C, \\ (a,b) &\to (a+1,b-2) && \text{with weight } 1, \end{aligned}$$

   which stay weakly above the x-axis.



2. *The sum of the monomials* $C^{n_2(T)}$ *over all 2-3–trees* T *on* $n+1$ *nodes, where* $n_2(T)$ = *number of nodes of* T *with 2 children.*

3. *The coefficient of* x^n *in* $\frac{1}{n+1}(1+Cx^2+x^3)^{n+1}$

4. *For* $C \geq 3$, *the integral moment*

$$\int_{t_2}^{t_1} t^n w(t)dt$$

where $t_2 < t_1$ *are the two larger roots of the discriminant of the cubic* z^3+Cz^2-tz+1, *and* $w(t) = w_C(t)$ *is positive for* $t_2 < t < t_1$.

5. *For* $C = 3$, *the integral moment*

$$\frac{\sqrt{3}}{2\pi}3^{n+4}\int_0^1 f^n(u)g(u)du, \qquad \text{where}$$

$$\begin{aligned} f(u) &= 9u(1-u)-1 \\ g(u) &= u^{\frac{1}{3}}(1-u)^{\frac{2}{3}}(1-2u) \end{aligned}$$

- The leading coefficient in $\mu_n(C)$ is the Catalan number $\frac{1}{2n+1}\binom{2n+1}{n}$ for μ_{2n} $(n > 0)$, and the binomial coefficient $\binom{2n+1}{n+1}$ for μ_{2n+1}.
- For the critical value $C = 3$

  $\mu_0 = 1,\ \mu_1 = 0,\ \mu_2 = 3,\ \mu_3 = 1,\ \mu_4 = 18,\ \mu_5 = 15, \mu_6 = 138, \ldots$??

Hankel determinants

$\Delta_n = \Delta_n(C) = \det[\mu_{i+j}(C)]$:

$$\begin{aligned}
\Delta_0 &= 1 \\
\Delta_1 &= C \\
\Delta_2 &= -1 + C^3 \\
\Delta_3 &= -2 - 3C^3 + C^6 \\
\Delta_4 &= -14C - 6C^7 + C^{10}
\end{aligned}$$

For $C = t + 3$, $\Delta_n = \det[\mu_{i+j}(3 + t)]$

$$\begin{aligned}
\Delta_0 &= 1 \\
\Delta_1 &= 3 + t \\
\Delta_2 &= 26 + 27t + 9t^2 + t^3 \\
\Delta_3 &= 646 + 1377t + 1188t^2 + 537t^3 + 135t^4 + 18t^5 + t^6 \\
\Delta_4 &= 45885 + 166198t + 264627t^2 + 245430t^3 + 147420t^4 + \\
&\qquad 60102t^5 + 16884t^6 + 3234t^7 + 405t^8 + 30t^9 + t^{10}
\end{aligned}$$

Conjecture: *The polynomials Δ_n have nonnegative integral coefficients.*

Theorem: *The constant terms $1, 3, 26, 646, 45885, \ldots$ of Δ_n are given by the Robbins-Rumsey sequence*

$$RR(n) = \prod_{k=0}^{n} \frac{\binom{6k+4}{2k+2}}{2\binom{4k+3}{2k+2}}$$

$\Rightarrow$ Alternating Sign Matrices (ASMs) with vertical symmetry are involved.

On ASMs

An $n \times n$ matrix with entries from $\{-1, 0, 1\}$ is an Alternating Sign Matrix (ASM) if

1. Every row and column has sum 1,
2. In every row and column, the non-zero entries start with 1 and alternate in sign.

Every permutation matrix is an ASM, and for $n = 1, 2$ these are the only ASMs. For $n = 3$, there are 7 ASMs, the six 3×3 permutation matrices and the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

A general formula for the number of $n \times n$ ASMs was conjectured by Mills–Robbins–Rumsey to be

$$\prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}$$

and proved by Zeilberger. The number of $(2n+3) \times (2n+3)$ ASMs symmetric about a vertical axis starts out as $1, 3, 26, 646, 45885$ for $n = 0, 1, 2, 3, 4$. The formula

$$RR(n) = \prod_{k=0}^{n} \frac{\binom{6k+4}{2k+2}}{2\binom{4k+3}{2k+2}}$$

was conjectured by Robbins-Rumsey. A proof was recently announced by Kuperberg.

Hankel determinant $\Delta_n(C)$ at $C = 3$

Theorem: $\Delta_n(3) = \det[a_{i+j}]_{0\le i,j\le n}$, *where* $a_k = b_k(x) = \sum_{j=0}^{k} \frac{j+1}{k+1}\binom{3k-j+1}{k-j} x^j$ *for any* x. *In particular, we can take*

1. $a_k = b_k(0) = \frac{1}{k+1}\binom{3k+3}{k}$,
2. $a_k = b_k(1) = \frac{1}{k+1}\binom{3k+1}{k}$,
3. $a_k = b_k(2) = \binom{3k+2}{k}$.

The first few polynomials $b_k(x)$ *are*

$$
\begin{aligned}
b_0(x) &= 1 \\
b_1(x) &= x + 2 \\
b_2(x) &= x^2 + 4x + 7 \\
b_3(x) &= x^3 + 6x^2 + 18x + 30
\end{aligned}
$$

The $b_k(x)$ are the enumerators of Carlitz for certain 2–line arrays.

How to prove $\Delta_n(3) = RR(n)$?

- Take $a_k = b_k(2)$. $\Delta_n(3)$ is the determinant of
$$A_n = \left[\binom{3(i+j)+2}{i+j}\right]_{0\le i,j\le n}$$
- Find lower triangular, unit diagonal W with $WA_n = U$ is upper triangular (i.e. the LDU decompositon) with diagonal elements $\frac{\binom{6k+4}{2k+2}}{2\binom{4k+3}{2k+2}}$.
- "Guess" the nonzero elements of the matrix W:
$$w_{i,j} = \sum_k \left[\frac{(-1)^{i+j+k}(3k)!}{(2k+1)!k!}\binom{2i}{i+j+k}\frac{(3i+2)!(j+k)!(2i+2j+2k+1)!}{(3j+3k+2)!i!(4i+1)!}\right].$$
- Prove that
$$\sum_{i=0}^{n} w_{m,i}a_{i,n} = \begin{cases} 0 & \text{if } m < n \text{ (below the diagonal)} \\ \frac{\binom{6n+4}{2n+2}}{2\binom{4n+3}{2n+2}} & \text{if } m = n \text{ (on the diagonal)} \end{cases}$$

- Use automated methods of Wilf and Zeilberger.
- Kurt Wegschaider's Multisum Mathematica program to find certificates for the double sum identities

  Sub-diagonal identity

$$\sum_{i=0}^{n}\sum_{j=0}^{n-i}\frac{(-1)^{i+j}(3j)!(i+j)!(2+3i+3m)!(1+2i+2j+2n)!}{j!(1+2j)!(2+3i+3j)!(i+m)!(2+2i+2m)!(-i-j+n)!(i+j+n)!}=0$$

  Diagonal identity

$$\sum_{i=0}^{n}\sum_{j=0}^{n-i}\frac{(-1)^{n+i+j}(3j)!(3n+3i+2)!(i+j)!(2n+2i+2j+1)!(3n-1)!(4n+3)!}{j!(2j+1)!(n+i+j)!(n-i-j)!(n+i)!(2n+2i+2)!(3j+3i+2)!(n-1)!(2n+1)(6n+1)!}=1.$$

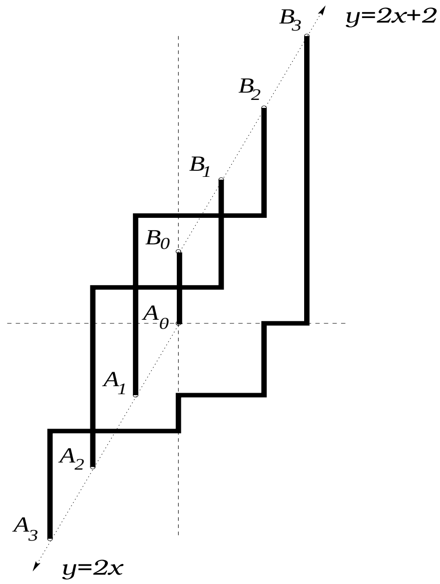
- Two "one-line proofs"
  - The certificate for Sub-diagonal: 4 lines
  - The certificate for Diagonal: over 20,000 lines, 1.2MB file.

`http://www.cs.ucsb.edu/~omer/diagonal-certificate.txt`.

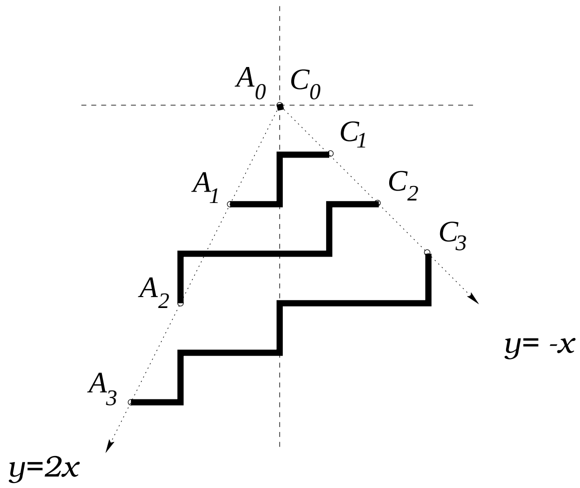
Path Interpretations: Hankel Paths

$$A_i = (-i, -ri), \text{ and } \quad B_i = (i, ri + r), \qquad (0 \le i \le n).$$

Path model for the $(n+1) \times (n+1)$ Hankel matrix $\left[\binom{(r+1)(i+j)+r}{i+j}\right]$.



Theorem: *For any integer* $r \geq 1$, $\det\left[\binom{(r+1)(i+j)+r}{i+j}\right]_{0\leq i,j\leq n}$ *is the number of non-intersecting tuples of paths* $\Pi = (\pi_0, \pi_1, \ldots, \pi_n)$ *with* $\pi_i : A_i \to C_i$ *where* $A_i = (-i, -ri)$, *and* $C_i = (i, -i)$, $(0 \leq i \leq n)$

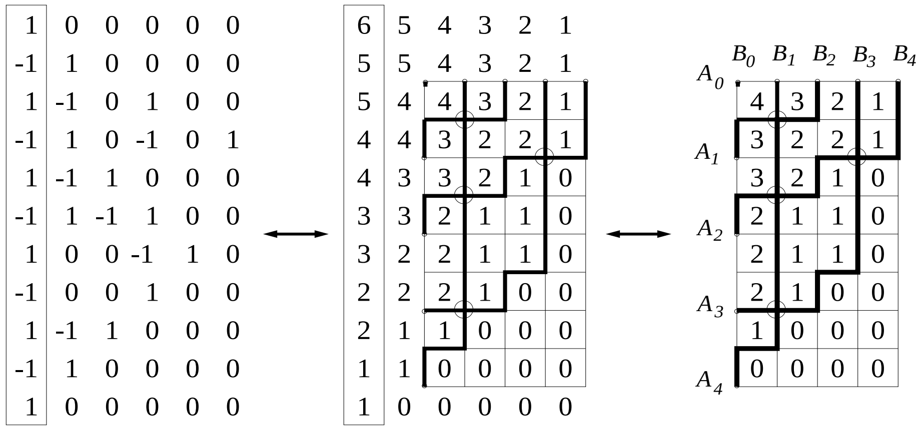


Hankel paths for $r = 2$ and $n = 3$.

$\Delta_n(3)$ enumerates $(n+1)$-tuples of these paths for $r = 2$.

Path Interpretations: ASMs

Robbins and Rumsey's corner-sum matrix $\Leftrightarrow$ osculating path model for ASM with vertical symmetry: $(n+1)$-tuples of osculating paths $\Pi = (\pi_0, \pi_1, \ldots, \pi_n)$ where $A_i = (0, -2i)$, $\quad B_i = (i, 0)$, and $\pi_i : A_i \to B_i$ for $i = 0, 1, \ldots, n$.



11×11 vertically symmetric ASM, the corner sum matrix, and the corresponding 5-tuple of osculating paths.

Tableau Interpretations

Call a tableau (or a shifted tableau) with n rows of odd staircase shape $\lambda = (2n-1, \ldots, 3, 1)$ good if

1. The entries are bounded by n,
2. The first cell in row i is labeled i,
3. The entries are weakly-increasing down the anti-diagonals.

$$T_1 = \begin{array}{ccccc} 1 & 1 & 1 & 2 & 2 \\ 2 & 3 & 3 & & \\ 3 & & & & \end{array} \qquad T_2 = \begin{array}{ccccc} 1 & 1 & 1 & 2 & 2 \\ & 2 & 2 & 3 & \\ & & 3 & & \end{array}$$

T_1: a column-strict good tableau of shape (5,3,1) (Hankel paths)
T_2: a shifted good tableau of shape (5,3,1) (Osculating paths)

11111	11111	11111	11112	11112	11112	11113	11113	11113	11122	11122	11122	11123
222	223	233	222	223	233	222	223	233	222	223	233	222
3	3	3	3	3	3	3	3	3	3	3	3	3

11123	11123	11133	11133	11222	11222	11223	11223	11233	11233	12222	12223	12233
223	233	223	233	223	233	223	233	223	233	233	233	233
3	3	3	3	3	3	3	3	3	3	3	3	3

Column-strict good tableaux of shape $(5, 3, 1)$

11111	11111	11111	11112	11112	11112	11113	11113	11122	11122	11122	11123	11123
222	223	233	222	223	233	223	233	222	223	233	223	233
3	3	3	3	3	3	3	3	3	3	3	3	3

11133	11222	11222	11222	11223	11223	11233	12222	12222	12222	12223	12223	12233
233	222	223	233	223	233	233	222	223	233	223	233	233
3	3	3	3	3	3	3	3	3	3	3	3	3

Shifted good tableaux of shape $(5, 3, 1)$

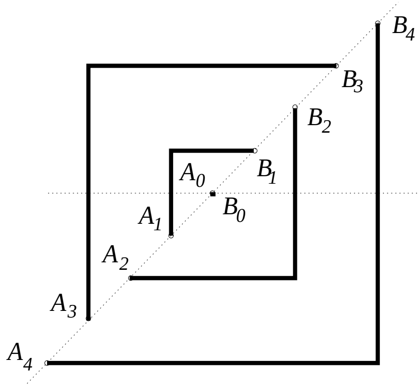
Bijection between the two sets of good tableaux $\Rightarrow$ bijective proof of the Rumsey-Robbins formula for ASM with vertical symmetry.

Further Results/Conjectures

Easy path models for other Hankel determinants:

Theorem: $\det \left[\binom{2(i+j)}{i+j} \right]_{0 \le i,j \le n} = 2^n$.

Points: $A_i = (-i, -i)$, and $B_i = (i, i)$, $(0 \le i \le n)$. Determinant enumerates all $(n+1)$-tuples of paths $\Pi = (\pi_0, \pi_1, \ldots, \pi_n)$ corresponding to the identity permutation in which each π_i is a *right-angle*.



There is only a single $(n+1)$-tuple of Hankel paths (horizontal) for $\det\left[\binom{(r+1)(i+j)+r}{i+j}\right]_{0\le i,j\le n}$ with $r=1$.

Since

$$\det\left[\frac{1}{2(i+j)+1}\binom{2(i+j)+1}{i+j}\right] = \det\left[\binom{2(i+j)+1}{i+j}\right]$$

This gives a combinatorial proof of the following well-known result on the Catalan-Hankel determinant:

Theorem: $\det[C_{i+j}]_{0\le i,j\le n} = 1$, *where* $C_k = \frac{1}{2k+1}\binom{2k+1}{k}$.

The following should have a combinatorial proof by q-weighting Hankel paths with the standard q-analogue

$$C_n(q) = \sum_{k=0}^{n-1} q^k C_k(q) C_{n-1-k}(q), \qquad (C_0(q) = 1)$$

Conjecture: $\det[C_{i+j}(q)]_{0\le i,j\le n} = q^{\frac{(n-1)n(4n+1)}{6}}$

Values of $\det\left[\binom{(r+1)(i+j)+r}{i+j}\right]_{0\le i,j\le n}$ for small r and n:

$r\backslash n$	0	1	2	3	4
1	1	1	1	1	1
2	1	3	26	646	45885
3	1	6	206	40083	44042301
4	1	10	950	848465	7096349476
5	1	15	3200	9604260	403895099151

1, 6, 206, 40083, 44042301, ... are the values of the Hankel determinant $\Delta_n = \det[\mu_{i+j}]_{0\le i,j\le n}$, where μ_k is the moments of the linear functional defined by the 4-term recursion at the critical value $C = 6$.

Conjecture: *For $a \ge 4$, the values of the Hankel determinants $det[h_{i+j}]_{0,\le i,j\le n}$ for $h_k = \binom{ak+b}{k}$ are never smooth.*

We take the moments $\mu_n = \mu_n(C)$ with $C = 3$ with

$$\mu_n = 3^{n+4}\frac{\sqrt{3}}{2\pi}\int_0^1 f^n(v)g(v)dv$$

where $f(u) = 9u(1-u) - 1,\ g(u) = u^{\frac{1}{3}}(1-u)^{\frac{2}{3}}(1-2u)$. Using this, we obtain an expression for the corresponding Hankel determinants $\Delta_n = \det[\mu_{i+j}]_{0\le i,j\le n}$ for $C = 3$ as a Macdonald-type integral.

Theorem:

$$\Delta_n(3) = (\frac{\sqrt{3}}{2\pi})^{n+1}\frac{3^{(n+1)(3n+4)}}{(n+1)!}\int_{I^{(n+1)}}\prod_{0\le i<j\le n}(v_i - v_j)^2\prod_{i=0}^{n} g(u_i)du_i$$

where

$$v_i = u_i(1-u_i)$$

ASMs Symmetric wrt both Axes

Similar to the standard q-analogue of the Catalan numbers C_k, consider the q-analogue of $T_k = \frac{1}{3k+1}\binom{3k+1}{k}$:

$$T_n(q) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1-i} q^{i+2j} T_k(q) T_j(q) T_{n-1-i-j}(q), \qquad (T_0(q) = 1)$$

Let $\Delta_n(q) = \det[T_{i+j}(q)]_{0 \le i,j \le n}$.

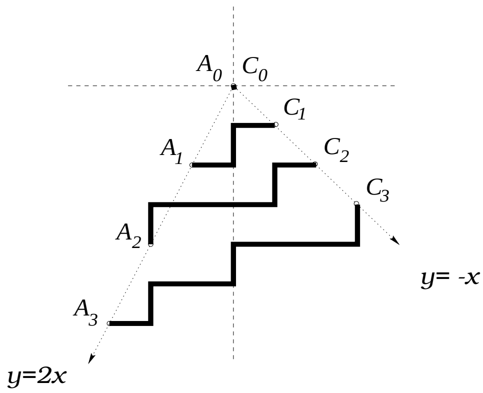
Conjecture: *$\Delta_n(q)$ has nonnegative integral coefficients.*

Conjecture: *$\Delta_n(-1)$ is the number of $(2n+1) \times (2n+1)$ ASMs symmetric with respect to both vertical and horizontal axes.*

Note: $T_k(-1)$ is the difference of two binomial coefficients.

An (easy) Binomial Identity from Hankel Paths

The proof that $\det\left[\binom{(r+1)(i+j)+r}{i+j}\right]_{0\le i,j\le n}$ enumerates non-intersecting tuples $\Pi = (\pi_0, \pi_1, \ldots, \pi_n)$ with $\pi_i : A_i \to C_i$:



Uses the (trivial) binomial identity

$$\binom{rk}{k} = r\binom{rk-1}{k-1}.$$

What is a direct/elementary/obvious bijective proof of this?

Print versions

- "Polynomial Families Satisfying a Riemann Hypothesis," Congressus Numerantium, 2001.
- "From a Polynomial Riemann Hypothesis to Alternating Sign Matrices," The Electronic Journal of Combinatorics, 2001.

Electronic versions

http://www.cs.ucsb.edu/˜omer