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Generalized permutation patterns and a classification of Euler-Mahonian statistics

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(respectively)

A *pattern-function* is a function $p : \mathcal{S}_n \to \mathbb{N}$ that counts the number of *occurrences* of a certain pattern in a permutation π.

$(a\text{-}b\text{-}c)$ counts increasing subsequences of length 3 in a permutation.

The absence of a dash between two letters in a pattern says that the corresponding letters must be adjacent in π.

$$(b\text{-}ca)\ 3\,4\,1\,5\,2\ =\ 3$$

3 – 41

3 – 52

4 – 52

Two well-known patterns:

$$(b\text{-}a)\pi = \mathsf{INV}\pi \qquad (ba)\pi = \mathsf{des}\,\pi$$

The "classical" study of *pattern avoidance* deals with patterns of type $(x\text{-}y\text{-}z)$ and permutations such that $(x\text{-}y\text{-}z)\pi = 0$.

The *major index* of a permutation is the sum of (the positions of) its descents. For example,

$$\mathrm{MAJ}\,4\,6\,1\,5\,2\,3 \;=\; 2+4=6.$$

Thus, MAJ counts, for each descent (ba), the number of letters in π preceding the a. Hence,

$$\mathrm{MAJ} = (a\text{-}cb) + (b\text{-}ca) + (c\text{-}ba) + (ba).$$

MAJ is *mahonian* i.e., it has the same distribution as INV.

Another mahonian statistic is MAK, which can be defined thus:

$$\mathrm{MAK} = (b\text{-}ca) + \underbrace{(cb\text{-}a) + (a\text{-}cb) + (ba)}_{\text{sum of descent bottoms}}.$$

The mahonian statistic MAD is obtained from MAK by replacing *descent bottoms* with *descent differences*:

$$\mathrm{MAD} = (b\text{-}ca) + \underbrace{(b\text{-}ca) + (ca\text{-}b) + (ba)}_{\text{sum of descent differences}}.$$

Most (all?) mahonian permutation statistics in the literature can be written as combinations of patterns, or else can be translated into such combinations.

Example: DEN corresponds to MAK in a way that translates excedances into descents, excedance bottoms into descent bottoms etc.

Why do all the mahonian statistics above consist of three 3-patterns with one dash and the 2-pattern (ba)?

Definition. The *weight on $\mathcal{S}_n$* of a function f is the sum $\sum_{\pi \in \mathcal{S}_n} f(\pi)$.

Lemma. *The weight on $\mathcal{S}_n$ of a d-pattern with $k-1$ dashes is given by*

$$\frac{n!}{d!}\binom{n+k-d}{k}.$$

The weight of $(b-a)$, the number of inversions, on $\mathcal{S}_n$, and thus the weight of any mahonian function, is

$$\binom{n}{2} \cdot \binom{n}{2}(n-2)! = \frac{n!}{2}\binom{n}{2}.$$

In this and the next two slides, we will be counting the implicit dashes at the beginning and end of a pattern, and allowing patterns that do not have a dash at the beginning or end.

In this setup, the pattern $(b\text{-}a\text{-})$ counts the number of inversions involving the first letter of a permutation.

Because $\binom{n}{2}$ is a polynomial in n of degree 2, it can be shown that a mahonian linear combination of patterns cannot have any patterns with more than three dashes.

We need to find all possible combinations of d-patterns with 0, 1, 2 or 3 dashes such that the weight is the same as that of the number of inversions.

If the respective numbers of these patterns are x, y, z and w then they must satisfy the following equation, where $m = n - d$:

$$\binom{m-1}{-1}x + \binom{m}{0}y + \binom{m+1}{1}z + \binom{m+2}{2}w$$

$$= \frac{n!}{2}\binom{n}{2}$$

for all $n \geq d$.

Solving this (for $n = d, \dots, d+3$) is equivalent to solving a system of linear equations corresponding to the following matrix.

$$\begin{matrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & 6 \\ 0 & 1 & 4 & 10 \end{matrix}$$

There is a unique solution.

Using linear algebra we can show:

Theorem. *Let f be a combination of patterns of length at most d. If f is mahonian then f can be written as a sum of $k!/2$ k-patterns with no dashes, for $2 \leq k < d$, and $d!/2$ d-patterns with one dash.*

In particular, mahonian statistics with patterns of length ≤ 3 must consist of (ba) and three 3-patterns with one dash.

There are essentially 28 different such mahonian statistics. Two of these are equal (as functions) to $(b\text{-}a) = \mathrm{INV}$.

Saying that (des, S) and (des, T) are equidistributed, where S and T are two 3-statistics, means that their sums of 3-patterns are equidistributed.

$(\mathrm{des}, \mathrm{MAJ})$ and $(\mathrm{des}, \mathrm{MAK})$ are equidistributed because these are:

$$(a\text{-}cb) + (b\text{-}ca) + (c\text{-}ba)$$

$$(a\text{-}cb) + (b\text{-}ca) + (cb\text{-}a)$$

Define an equivalence relation on the set of mahonian 3-statistics by setting $S \sim T$ if (des, S) and (des, T) are equidistributed.

There are 7 equivalence classes.

1	$(ac - b)$	$+$	$(ac - b)$	$+$	$(b - ac)$	
2	$(ac - b)$	$+$	$(ac - b)$	$+$	$(b - ca)$	
3	$(ac - b)$	$+$	$(b - ca)$	$+$	$(b - ca)$	
4	$(b - ca)$	$+$	$(b - ca)$	$+$	$(ca - b)$	MAD
5	$(ac - b)$	$+$	$(ba - c)$	$+$	$(cb - a)$	stat
6	$(ac - b)$	$+$	$(ba - c)$	$+$	$(c - ba)$	???
7	$(a - cb)$	$+$	$(b - ca)$	$+$	$(cb - a)$	MAK
8	$(a - cb)$	$+$	$(b - ca)$	$+$	$(c - ba)$	MAJ
9	$(a - cb)$	$+$	$(ca - b)$	$+$	$(cb - a)$	MAKℓ
10	$(ac - b)$	$+$	$(ca - b)$	$+$	$(cb - a)$	stat$'$
11	$(a - cb)$	$+$	$(b - ca)$	$+$	$(b - ca)$	FZ 2000
12	$(a - cb)$	$+$	$(c - ab)$	$+$	$(c - ba)$	stat$''$
13	$(bc - a)$	$+$	$(ca - b)$	$+$	$(ca - b)$	FZ 2000
14	$(bc - a)$	$+$	$(ca - b)$	$+$	$(cb - a)$	INV

All Mahonian 3-functions (omitting (ba))
up to trivial bijections

The first four belong to those defined
by Simion and Stanton

As far as we know, the only mahonian statistic in the literature with patterns of length > 3 is one defined by Haglund and arising from rook theory. His statistic can be written in terms of excedances and then translated into a sum of

twelve 4-patterns with 1 dash,

three 3-patterns with no dash,

and (ba)

$$
\begin{aligned}
&(ba) + (cab) + (cba) + (acb) \\
&+2 \cdot (ca\text{-}db) + 2 \cdot (cb\text{-}da) \\
&+(dc\text{-}ab) + (dc\text{-}ba) + (da\text{-}bc) + (db\text{-}ac) \\
&+(ad\text{-}cb) + (ac\text{-}db) + (ab\text{-}dc) + (ba\text{-}dc)
\end{aligned}
$$

The number of permutations in $\mathcal{S}_n$ avoiding $(a\text{-}bc)$ is the Bell number, i.e. number of partitions of an n-set.

Let

$$S = \{\pi \in \mathcal{S}_n \mid (a\text{-}bc)\pi = 0 \text{ and } (b\text{-}ac) = 1\}.$$

Then

$$\begin{aligned} \#S &= \text{genus of the graph of the } (n+1)\text{-cube} \\ &= \text{betti number of the } (n-1)\text{-cube} \end{aligned}$$

Some other examples of numbers/structures appearing as the number of permutations with prescribed values for certain patterns:

Set partitions
Permutations by number of cycles
Involutions in $\mathcal{S}_n$
Catalan numbers
Bessel numbers
Motzkin numbers
Directed animals
Various ballot numbers
Chebyshev polynomials
Columns from expansion of $(1+x+x^2)^n$.
GF's of type

$$(1-ax)(1-bx)\cdots/(1-cx)(1-dx)\cdots.$$

$$\sum_{\pi \in \mathcal{S}_\infty} x^{(ab)} y^{(ba)} p^{(b-ca)} q^{(ca-b)} t^{(a)} =$$

$$\cfrac{1}{1 - \cfrac{x\,t}{1 - \cfrac{y\,t}{1 - \cfrac{x\,[2]\,t}{1 - \cfrac{y\,[2]\,t}{1 - \cfrac{x\,[3]\,t}{1 - \cfrac{y\,[3]\,t}{\ddots}}}}}}}$$

where

$$[k] = p^{k-1} + p^{k-2}q + \cdots + pq^{k-2} + q^{k-1}$$

$$\sum_{\pi\in\mathcal{S}_\infty} q^{(b-a)} t^{(a)} \ =$$

$$\cfrac{1}{1-\cfrac{t}{1-\cfrac{q\,t}{1-\cfrac{q\,[2]\,t}{1-\cfrac{q^2\,[2]\,t}{1-\cfrac{q^2\,[3]\,t}{\ddots}}}}}}$$

where

$$[k] = 1 + q + \cdots + q^{k-1}$$

$$\sum_{\substack{\pi\in\mathcal{S}_\infty \\ (a\text{-}c\text{-}b)\,\pi = 0}} x_0^{(a)}\, x_1^{(a\text{-}b)}\, x_2^{(a\text{-}b\text{-}c)} \cdots t^{(a)} \;=$$

$$\cfrac{1}{1-\cfrac{x_0\,t}{1-\cfrac{x_0\,x_1\,t}{1-\cfrac{x_0\,x_1^2\,x_2\,t}{1-\cfrac{x_0\,x_1^3\,x_2^3\,x_4\,t}{1-\cfrac{x_0\,x_1^4\,x_2^6\,x_3^4\,x_4\;\;t}{\ddots}}}}}}$$

The $(n+1)$st denominator is $\prod_{k\geq 0} x_k^{\binom{n}{k}}$