Ramsey theory and fast growing functions.

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Part I

Ramsey's theorem

If A is a set then $[A]^k$ denotes the set of all k-element subsets of A, $[n] = \{1, 2, \ldots, n\}$ and $[n]^k = [[n]]^k$.

Theorem 1 (Ramsey's theorem, infinite version). *If $\chi : [\mathbb{N}]^k \to [r]$ is a colouration then there exists an infinite $A \subseteq \mathbb{N}$ such that $[A]^k$ is monochromatic, i.e. there is an $i \in [r]$ such that $[A]^k \subseteq \chi^{-1}(\{i\})$.*

Proof. We prove it by induction over k. For $k = 1$ the proof is trivial. Assume the theorem holds for k and we prove it for $k + 1$. Let $\chi : [\mathbb{N}]^{k+1} \to [r]$. We construct a sequence of infinite sets S_i and elements of those sets $x_i \in S_i$. Let $S_0 = \mathbb{N}$, if S_i is chosen choose $x_i \in S_i$ arbitrarily and let

$$\chi' : [S_i - \{x_i\}]^k \to [r], \chi'(u) = \chi(u \cup \{x_i\}).$$

Take S_{i+1} to be an infinite subset of $S_i - \{x_i\}$ such that $[S_{i+1}]^k$ is monochromatic (we get this set from the induction hypothesis).
We see that $S_i \supseteq S_j$ if $i \leq j$ so that

$$\chi(\{x_i, x_{i_1}, \ldots, x_{i_k}\}) = \chi(\{x_i, x_{i'_1}, \ldots, x_{i'_k}\})$$

if $i < i_1 < \ldots < i_k$ and $i < i'_1 < \ldots < i'_k$. Now we can partition the set $\{x_1, x_2, \ldots\}$ into a finite number of blocks

$$T_j = \{x_i \mid \chi(\{x_i, x_{i_1}, \ldots, x_{i_k}\}) = j, i < i_1 < \ldots < i_k\}.$$

Let A be an infinite block of this partition. It is now clear that if $\mathbf{x}, \mathbf{x}' \in [A]^{k+1}$ then $\chi(\mathbf{x}) = \chi(\mathbf{x}')$ so the set $[A]^{k+1}$ is monochromatic. $\square$

Definition 2. Let $n \to (l)^k_r$ be the relation that holds iff for every r-colouring of $[n]^k$ there exists a set $A \subseteq [n]$ such that $|A| = l$ and $[A]^k$ is monochromatic.

Theorem 3 (Ramsey's theorem, finite version). *For all natural numbers k, l, r there exists n such that $n \to (l)^k_r$.*

To prove this we will use a compactness argument but first we have to define hypergraphs.

Definition 4. An hypergraph H consists of a set of vertices $V(H)$ and a set $E(H)$ of subsets of $V(H)$ called the edges. An r-colouring of an hypergraph H is a map $\chi : V(H) \to [r]$ such that no $X \in E(H)$ is monochromatic. $\chi(H)$ is the least r such that there exists an r-colouring of H.

If $W \subseteq V(H)$ then H_W is the hypergraph such that $V(H_W) = W$ and $E(H_W) = E(H) \cap \mathcal{P}(W)$.

Theorem 5 (Compactness theorem). *If H is an hypergraph where all the edges $X \in E(H)$ are finite and for all finite $W \subseteq V(H)$ $\chi(H_W) \leq r$ then $\chi(H) \leq r$.*

Proof. We prove the case when $V(H)$ is countable, for simplicity we assume $V(H) = \mathbb{N} - \{0\}$.
Let χ_n be an r-colouring of $H_{[n]}$, i.e. a map $[n] \to [r]$. We now construct an r-colouring χ of H by induction. Suppose that $\chi(1), \chi(2), \ldots, \chi(j)$ have been defined such that the set

$$S_j = \{n \mid n \geq j \text{ and } \chi(i) = \chi_n(i) \text{ for all } 1 \leq i \leq j\}$$

is infinite (the base case is trivial). For some colour c the set $T = \{n \in S_j - \{j\} \mid \chi_n(j+1) = c\}$ is infinite, let $\chi(j+1)$ be that colour c. It is quite easy to see that $S_{j+1} = T$, so S_{j+1} is infinite.
Let $X = \{x_1, \ldots, x_m\} \in E(H)$ and $M = \text{Max}(X)$, it is clear that there exists $n \geq M$ such that $\chi_n(i) = \chi(i)$ for all $i \leq n$ (since $S_M \neq \emptyset$). Since χ_n is an r-colouring, X is not monochromatic under χ_n and thus not under χ. □

Proof of Ramsey's theorem, finite version. Suppose that the theorem is false for some r, k, l. Let $V(H) = [\mathbb{N}]^k$ and $E(H) = \{[A]^k \mid A \subseteq \mathbb{N} \text{ and } |A| = l\}$, then by the assumption $\chi(H_W) \leq r$ for all finite $W \subseteq V(H)$. By the compactness theorem $\chi(H) \leq r$ but this contradicts the infinite version of Ramsey's theorem, since this gives us an infinite set A such that $[A]^k$ is monochromatic. □

Other theorems

Theorem 6 (Van der Waerden's theorem). *For all $l, r \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that if $[n]$ is r-coloured there exist $a, d \in \mathbb{N}$ such that the set*

$$\{a + xd \mid 0 \leq x \leq l - 1\}$$

is monochromatic.

C_t^n is the n-dimensional cube over t elements, i.e.

$$C_t^n = \{(x_1, \dots, x_n) \mid x_i \in [t]\}.$$

A line in C_t^n is a set $\{\mathbf{x_1}, \dots, \mathbf{x_t}\} \subseteq C_t^n$ where $\mathbf{x_i} = (x_{i1}, \dots, x_{in})$ such that for all $1 \leq j \leq n$ either $x_{1j} = x_{2j} = \dots = x_{tj}$ or $x_{sj} = s$ for all $1 \leq s \leq t$.

Theorem 7 (Hales-Jewett theorem). *For all $r, t \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$, if C_t^n is r-coloured then there exists a monochromatic line.*

Part II

Definition 8. A set $L \subseteq \mathbb{N}$ is *large* if $|L| \geq \min(L)$.
$n \overset{*}{\rightarrow} (l)_r^k$ holds iff for every r-colouring of $[n]^k$ there exists a large set $A \subseteq [n]$ such that $|A| \geq l$ and $[A]^k$ is monochromatic.

Theorem 9 (Ramsey's theorem, PH version). *For all k, l, r there exists n such that $n \overset{*}{\rightarrow} (l)_r^k$.*

Proof. Suppose that the theorem is false for some r, k, l. Let $V(H) = [l, \infty)^k$ and $E(H) = \{[A]^k \mid A \subseteq [l, \infty) \text{ and } A \text{ large}\}$, then by the assumption $\chi(H_W) \leq r$ for all finite $W \subseteq V(H)$. By the compactness theorem $\chi(H) \leq r$ but this contradicts the infinite version of Ramsey's theorem, since this gives us an infinite set $A \subseteq [l, \infty)$ such that $[A]^k$ is monochromatic, so we could choose B as the first $\min(A)$ elements of A, then B will be a monochromatic edge. $\square$

Ordinal numbers and the extended Grzegorczyk hierarchy

The natural numbers is a sequence that is closed under the successor operation. Sometimes it could be nice to have a sequence that is closed under

limits as well, this gives us the ordinal numbers. They are a sequence extending the natural numbers, ω is the first ordinal bigger than all natural numbers and $\omega+1$ is the first ordinal bigger than ω and so on.
We could define the natural numbers to be $n = \{0, 1, \ldots, n-1\}$, in exactly the same way we can define the ordinal numbers to be the set of all smaller ordinal numbers, i.e. $\alpha = \{\beta \mid \beta < \alpha\}$, e.g. $\omega = \mathbb{N}$. This will give us a well-ordered sequence that is closed under unions and successor. The nice property about ordinal numbers is that every well-ordered set is order-isomorphic to a unique ordinal.

If α, β are ordinals then $\alpha + \beta$ is the ordinal "corresponding" to what we get if we put all elements in α "in front" of all the elements in β, e.g. $\omega + 1 = \mathbb{N} \cup \{\mathbb{N}\}$ but $1 + \omega = \omega$. The product $\alpha\beta$ is defined to be the ordinal corresponding to the well-order of $\alpha \times \beta$ with reverse lexicographic order, e.g. $\omega 2 = \omega + \omega$ but $2\omega = \omega$. Now we can define α^β in the following way

$$\alpha^{\beta+1} = \alpha^\beta \alpha$$
$$\alpha^\beta = \bigcup_{\gamma < \beta} \alpha^\gamma \text{ if } \beta \text{ is a limit ordinal.}$$

Define $\omega_0 = \omega$ and $\omega_{k+1} = \omega^{\omega_k}$ and let

$$\epsilon_0 = \lim_{k \to \infty} \omega_k \stackrel{def}{=} \bigcup_{k \in \mathbb{N}} \omega_k.$$

Theorem 10 (Cantor's normal form). *If $\alpha < \epsilon_0$ then α can be written as*

$$\alpha = \omega^{\beta_1} + \omega^{\beta_2} + \ldots + \omega^{\beta_l}$$

$\alpha > \beta_1 \geq \beta_2 \geq \ldots \geq \beta_l$*, where* $\beta_1, \beta_2, \ldots, \beta_l$ *are unique.*

Definition 11. If $\alpha = \omega^{\beta_1} + \omega^{\beta_2} + \ldots + \omega^{\beta_l}$, $\alpha > \beta_1 \geq \beta_2 \geq \ldots \geq \beta_l$ then we define

$$\alpha(n) = \omega^{\beta_1} + \omega^{\beta_2} + \ldots + \omega^{\beta_{l-1}} + \begin{cases} \omega^\gamma \cdot n & \text{if } \beta_l = \gamma + 1 \\ \omega^{\beta_l(n)} & \text{if } \beta_l \text{ is a limit ordinal} \\ \omega^{\beta_l} & \text{if } \beta_l = 0 \end{cases}$$

By induction it can easily be shown that

$$\lim_{n \to \infty} \alpha(n) = \bigcup_{n \in \mathbb{N}} \alpha(n) = \alpha.$$

Definition 12. We define functions $F_\alpha : \mathbb{N} \to \mathbb{N}$ for all $\alpha \le \epsilon_0$ by induction as follows

$$\begin{aligned} F_0(n) &= n+1, \\ F_{\alpha+1}(n) &= F_\alpha^{(n)}(n) = F_\alpha(F_\alpha(\ldots F_\alpha(n)\ldots)) \text{ iterated } n \text{ times}, \\ F_\alpha(n) &= F_{\alpha(n)}(n), \text{ if } \alpha \text{ is a limit ordinal.} \end{aligned}$$

Define $F_{\epsilon_0}(n) = F_{\omega_n}(n)$, where $\omega_0 = \omega$ and $\omega_{n+1} = \omega^{\omega_n}$.

Observe that F_ω is the Ackermann function, i.e. the "first" function that is not primitve recursive.

Theorem 13. *For all* $\alpha \le \epsilon_0$

1. F_α *is strictly increasing.*
2. *if* $\beta < \alpha$ *then* F_β *is majorized by* F_α, *i.e. there is* $c \in \mathbb{N}$ *such that for all* $x \ge c$ $F_\beta(x) < F_\alpha(x)$.

The Paris-Harrington theorem

Definition 14. Let the function $\mathrm{RM} : \mathbb{N}^3 \to \mathbb{N}$ be defined so that $\mathrm{RM}(l,k,r)$ is the smallest n such that $n \to (l)^k_r$ and let $\mathrm{PH} : \mathbb{N}^3 \to \mathbb{N}$ be defined so that $\mathrm{PH}(l,k,r)$ is the smallest n such that $n \overset{*}{\to} (l)^k_r$.

Theorem 15. $RM(l,k,r) \le F_3(Max(l,k,r)+1)$.

Lemma 16. *If* $f : \mathbb{N} \to \mathbb{N}$ *is a (description of a) provable recursive function, i.e. we can prove in PA that the function is total* ($PA \vdash \forall x \exists y f(x) = y$) *then* f *is majorized by* F_{ϵ_0}.

Theorem 17 (Paris-Harrington theorem). $\forall l,k,r \exists n(n \overset{*}{\to} (l)^k_r)$ *is not provable by finitistic methods, i.e.* $PA \nvdash \forall l,k,r \exists n(n \overset{*}{\to} (l)^k_r)$.

Suppose that it was provable in PA, then we have $\forall x \exists y y \to (x+1)^x_x$ so the function $f(x) = \mathrm{PH}(x+1,x,x)$ is provable recursive, by the above lemma f is majorized by F_{ϵ_0}.
The proof is now by showing that F_{ϵ_0} in fact is majorized by f.