Permutations avoiding arithmetic patterns

Peter Hegarty

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1. Subsets without k-term APs ($k > 2$)

- $A \subseteq \{1, ..., n\} := [n]$.

  $r_k(n) := \max |A|$.

- $A \subset \mathbf{N}$.

  $A(n) := \# A \cap [n]$,

$$\overline{d}(A) := \limsup_{n \to \infty} \frac{A(n)}{\xi(n)}.$$

Most natural choice : $\xi(n) := n$.

Upper asymptotic bound for $r_k(n)$

$$\Leftrightarrow$$

Upper asymptotic bound for $\overline{d}$

Lower asymptotic bound for $r_k(n)$

$$\Leftarrow$$

Lower asymptotic bound for $\overline{d}$

2. History

Conjecture (Szekeres, Erdos, Turan ('30s ?))

$$r_3(n) \sim n^{2/3},$$
$$r_3\left(\frac{1}{2}(3^k+1)\right) = 2^k$$

Salem and Spencer (1942)

$$r_3(n) \geq n^{1-\frac{\log 2+\epsilon}{\log\log n}}.$$

Behrend (1946)

$$r_3(n) \geq n^{1-\frac{c}{\sqrt{\log n}}}.$$

Roth (1952)

$$r_3(n) = O\left(\frac{n}{\log\log n}\right).$$

Take $\xi(n) = n$.

Szemeredi (1974) $\overline{d} = 0$ for all $k \geq 3$.

Corollary $r_k(n) = o(n)$ for all $k \geq 3$.

Szemeredi's theorem has since been reproven by Fürstenberg, using ergodic theory, and by Gowers using analytic methods.

3. Other arithmetic avoidance conditions

Definition A subset A of $[n]$ or $\mathbf{N}$ is called a *Sidon subset* if, whenever (a, b, c, d) is a quadruple of numbers from A such that $a+b=c+d$, then either $a=c, b=d$ or $a=d, b=c$.

$s(n) := \max|A|$.

Greedy algorithm : $|A| \sim n^{1/3}$.

Trivial upper bound : $s(n) \leq \sqrt{2n}$.

Erdos, Chowla (1944) There exist infinitely many n for which $s(n) > \sqrt{n}$. For all n, $s(n) > \sqrt{n} - cn^{5/16}$.

Ajtai, Komlos, Szemeredi(1981) There is a Sidon subset A of $\mathbf{N}$ such that
$A(n) > 10^{-3}(n \log n)^{1/3}$ for all $n >> 0$.

Kruckeberg (1961) There is a Sidon subset of **N** such that

$$\limsup_{n\to\infty} \frac{A(n)}{\sqrt{n}} \geq \frac{1}{\sqrt{2}}.$$

Erdos (1955) $\exists\, c > 0$ s.t. for any Sidon subset A of **N**,

$$\liminf_{n\to\infty} \frac{A(n)}{\sqrt{n/\log n}} \leq c.$$

Conjecture (Erdos) There exists an absolute constant $C > 0$ such that if A is a subset of $[n]$ with all subset sums distinct, then $|A| \leq \log_2 n + C$.

4. Pattern avoidance in permutations

Definition Let $n \geq k > 0$ integers. Let $\sigma \in S_k$, $\pi \in S_n$. π is said to *avoid* σ if there does not exist any k-tuple $(a_1, ..., a_k)$ such that

$$1 \leq a_1 < a_2 < \cdots < a_k \leq n,$$
$$\pi(a_i) < \pi(a_j) \Leftrightarrow \sigma(i) < \sigma(j).$$

5. Permutations avoiding arithmetic patterns

Definition Let S be an abelian semigroup. A permutation $\pi : S \to S$ is said to *avoid AP:s* if there does not exist any triple (a, b, c) of elements of S, not all equal, such that

$$a + c = b + b,$$
$$\pi(a) + \pi(c) = \pi(b) + \pi(b).$$

One one-page result in the literature.

Can make similar definition of *Sidon permutation* or *permutation with distinct subset sums*.

Theorem The greedy AP-avoiding mapping $\pi : \mathbf{N} \to \mathbf{N}$ is surjective and

$$\frac{1}{4} \leq \frac{\pi(n)}{n} \leq \frac{3}{2}.$$

Refinement

$$\frac{\pi(n)}{n} \geq \frac{3}{8}.$$

First Hopeless Conjecture

$$\lim_{n \to \infty} \frac{\pi(n)}{n} = 1.$$

Theorem Let G be an infinite abelian group. Then there exists an AP-avoiding permutation of G if and only if $G\backslash\Omega_2(G)$ has the same cardinality as G.

- Prove the result for all countable groups using a 'greedy' construction w.r.t. an initial well-ordering s.t. 'few' AP:s appear in any initial segment (proof also works for Sidon permutations)

- The following lemma + transfinite induction

  Lemma Let G be an abelian group, H a subgroup. Suppose there exist AP-avoiding permutations π_1, π_2 of H and G/H respectively. Then there exists an AP-avoiding permutation π of G. Moreover, π can be chosen so that $\pi(H) = H$ and $\pi|_H = \pi_1$.

Second Hopeless Conjecture There exists an AP-avoiding permutation of $\mathbf{Z}/n\mathbf{Z}$ for all n other than $n = 2, 3, 5, 7$.

Theorem For all n other than $n = 2, 3$ there exists a permutation of $\mathbf{Z}/n\mathbf{Z}$ avoiding 4-term AP:s.

Proposition 1 For no finite group G does there exist a Sidon permutation of G.

Proposition 2 If the finite group G possesses an AP-avoiding permutation, then

$$|\Omega_2(G)| \leq \sqrt{|G|}.$$