Simplicial complexes

- Are built from simplices, i.e. faces in different dimensions.
- If a face consisting of a set of vertices belongs to a complex, then all the subsets of this set are also included. Thus, a simplicial complex is determined by the set of maximal faces, w.r.t inclusion.
- Geometrically a 0-simplex is a point, a 1-simplex is an edge, a 2-simplex is a triangle and the 3-simplex is a tetrahedron.

CW-complexes

- The building blocks in CW-complexes are called cells.
- A d-cell is a topological space homeomorphic to B^d, the unit ball of dimension d in Euclidean space :

B^d ={x ° E^d: |x| 1}.

- The boundary of B^d is the unit (d-1)-sphere
  S^(d-1) ={x ° E^d: |x|=1}.

- Geometrically the point is a 0-
  cell, the line is a 1-cell, the disc is a 2-cell and a filled ball is a 3-cell.

•___ A CW-complex is built by successively attaching cells to an existing CW-complex. Attaching a cell means gluing its boundary to the old complex.

Morse functions

- Let ° be a d-simplex in a simplicial complex K. Let B(°) be the set of (d-1) faces of °. Let C(°) be the set of (d+1) simplices of K that contain °.

- Let f:KZ satisfying the following for all ° ° K

(1) f(°)f() for at most one ° C(°)
(2) f(°)f(°) for at most one ° ° B(a)
Then f is a **Morsefunction** on K.

Given a Morsefunction f a simplex ° is **critical** if :

(1) f(°)f() for all ° C(°)
(2) f(°)f(°) for all ° ° B(a)

- **Theorem:** Let K be a simplicial complex with a discrete Morse function. Let c_i be the number of critical simplices of dim i. Then K is homotopy equivalent to a CW-complex with c_i cells of dim i.

- **Definition:** For any simplicial complex K with a discrete Morse-function f and any real number c the level sub complex K(c) is the sub complex consisting of

all simplices ° of K so that f(°)<=c along with all of their faces.

Gradient vectorfields

- We can build a gradient vector field by drawing arrows as follows. Suppose ° is a non-critical simplex with ° ° B(°) satisfying f(°)<=f(°). Then we draw an arrow from ° to °.

- The definition of a Morse function implies that one of the two conditions for criticality must be true. This implies that every simplex ° satisfies **exactly** one of the following.

·.1. ° is the tail of exactly one arrow.
·.2. ° is the head of exactly one arrow.
·.3. ° is neither the head nor the tail of an arrow.

A gradient vectorfield V corresponds to a collection of pairs (°,°) such that

(1) ° ° B(°)
(2) f(°)f(°)

- **Definition:** A (discrete) vector field V on K is a collection of pairs (°,°) of simplices of K where

° °B(°) such that each simplex is in at most one pair of V.

- We say that a V-path is a sequence of simplices: $\circ_0, \circ_0, \circ_1, \circ_1\ \circ_2, \circ_2 \ldots\ \circ_r, \circ_{r+1}$ in a vectorfield V where °°B(°) and the pairs (°i,°i)°V . Such a path is non-trivial closed if r>=0 and °0= °r+1.

- **Theorem:** If V is gradient vectorfield of a Morse function the sequence $\circ_0, \circ_0, \circ_1, \circ_1\ \circ_2, \circ_2 \ldots\ \circ_r, \circ_{r+1}$ is a V-path iff:

  $f(\circ_0)f(\circ_0)>f(\circ_1)f(\circ_1)>f(\circ_2)f(\circ_2)..\ f(\circ_r)>f(\circ_{r+1})$.

  This is the same as saying we have to follow the arrows to get a V-path.

- **Theorem:** A discrete vector field V is the gradient vector field of a discrete Morse function if and only if there are no non-trivial closed V-paths.

A combinatorial point of view

- The faceposet P(k) of a simplicial complex K is the poset of faces of K w.r.t inclusion.
- Consider the Hasse diagram of P(k) as a directed graph with all arrows pointing down. Given a

vectorfield V on K if (°,°) °V (where ° ° B(°)) then flip the orientation of the edge (°,°) in P(k).

- **Fact:** There are no non-trivial closed V- paths in K if and only if there are no directed cycles in the Hasse diagram.

- A face of K is **paired** if it belongs to an up-arrow. Let u_p be the number of unpaired faces of dim p. Then from the main theorem of discrete Morse theory:

  K is homotopy equivalent to a CW-complex with exactly u_p cells of dimension p for each p.

Cancelling critical simplices

Theorem: Suppose f is a discrete Morse function on M such that °^p and ° ^(p+1) are critical and there is exactly one gradient path from B(°) to °. Then there is another Morse function g on M with the same critical simplices except that ° and ° are no longer critical.

- **My aim**
  In this talk I will give an introduction to what discrete Morse theory is and how to use it in topology and combinatorics. The theory is simple and there is not much you need to know before becoming a user. I will also show the advantage of translating a simplicial complex (I will give a short repition of this) to a CW-complex which I am soon going to give a definition of. One of the main reasons why we rather work with the CW-complex is that it can often be built by fewer cells (buildingblocks) than its related simplicial complex, without loss of the homology (topological information). I will return to this later on.