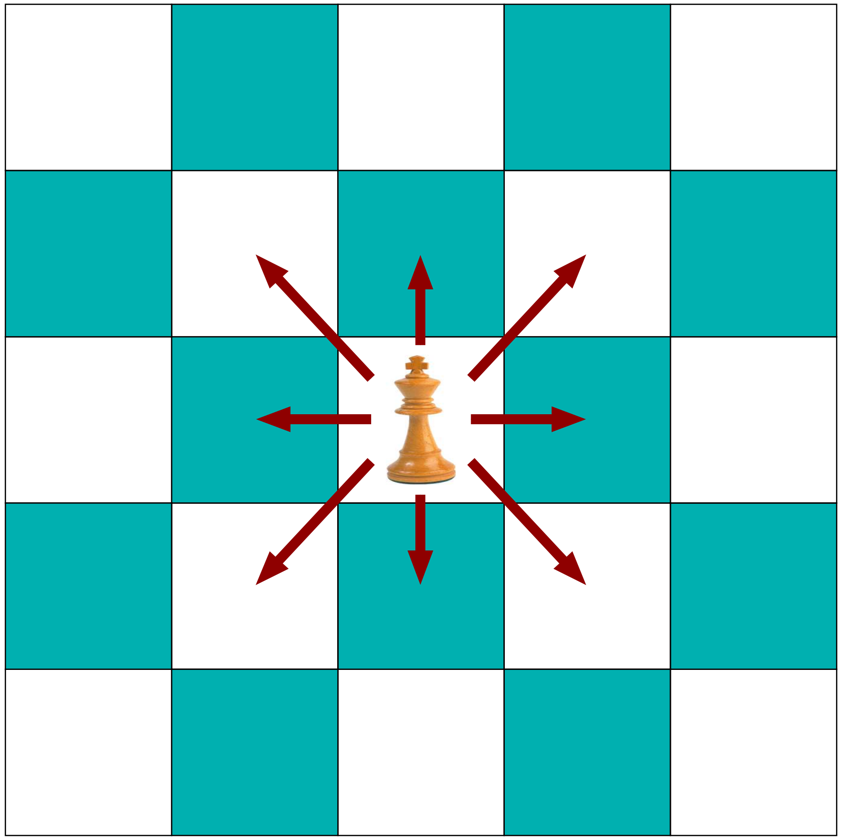
Non-attacking placements on chessboards

Sergey Kitaev

The king can attack one square in any direction.



H. Wilf: The problem of the kings.

On a $2m \times 2n$ chessboard, the maximum number of nonattacking kings that can be placed is mn, since each 2×2 cell can have at most one king.

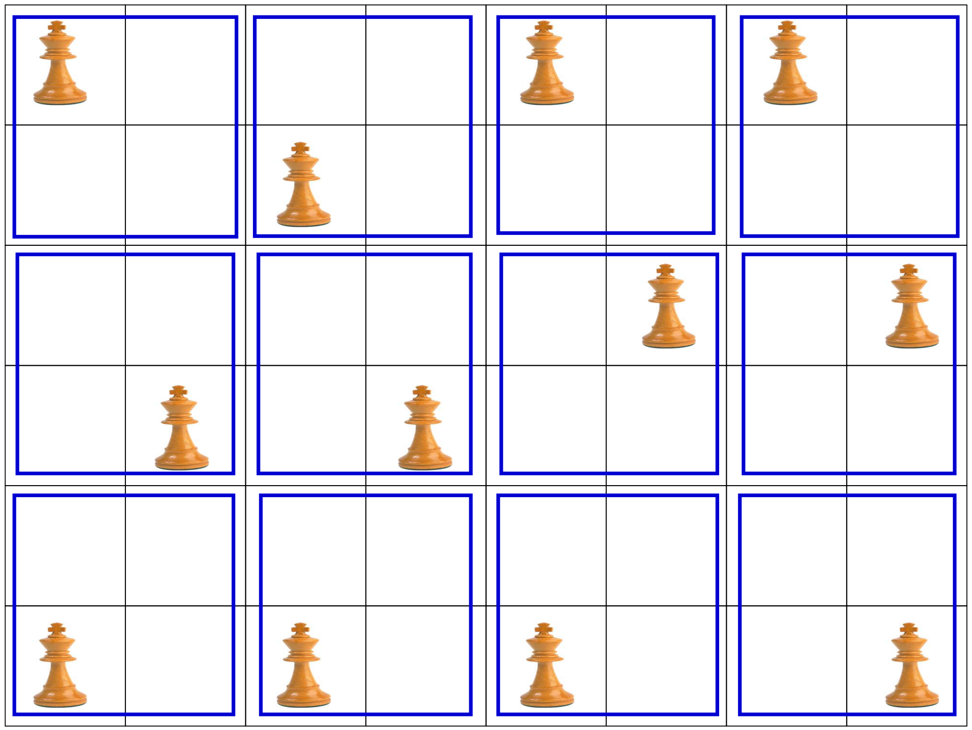
Let $f_m(n)$ be the number of ways that mn nonattacking kings can be placed on a $2m \times 2n$ chessboard.

Theorem. For each $m = 1, 2, 3, \ldots$ there are constants $c_m > 0$, d_m, and $0 \leq \theta_m < m + 1$ such that

$$f_m(n) = (c_m n + d_m)(m+1)^n + O(\theta_m^n) \qquad (n \to \infty).$$

For every maximal placement of kings, the chessboard is naturally divided into mn 2×2 cells, each containing exactly one king.

We say that a cell has type 1 (resp. 2,3,4) if the king sits in its NW (resp. NE, SE, SW) corner.



$$\longleftrightarrow \quad \begin{array}{cccc} 1 & 4 & 1 & 1 \\ 3 & 3 & 2 & 2 \\ 4 & 4 & 4 & 3 \end{array}$$

A matrix represents an allowable configuration of kings *iff* none of the following two letter words occurs:

$$21,\quad 24,\quad 31,\quad 34,\quad \begin{matrix}3\\1\end{matrix},\quad \begin{matrix}3\\2\end{matrix},\quad \begin{matrix}4\\1\end{matrix},\quad \begin{matrix}4\\2\end{matrix},\quad \begin{matrix}3 & \\ & 1\end{matrix},\quad \begin{matrix} & 4\\ 2 & \end{matrix}$$

Think of the columns of a matrix as wooden playing pieces, like dominoes.

If we scan one of the pieces from top to bottom we see a block of 0 or more 1's and 2's followed by a block of 3's and 4's. Hence the number of pieces is altogether

$$N(m) = (m+1)2^m.$$

Fix $m \geq 1$, and one of $N(m)$ pieces of height m, say the piece u $(1 \leq u \leq N(m))$.

Let $f_m(n, u)$ be the number of $m \times n$ arrays whose last (i.e., n-th) column is the given piece u, and $f_m(n) = \sum_u f_m(n, u)$. Then

$$f_m(n+1, u) = \sum_v f_m(n, v)\lambda(v \to u), \quad (f_m(1, u) = 1 \ (1 \leq u \leq N(m)))$$

in which λ is the truth value of the proposition that it is legal for the piece v to be immediately followed by the piece u.

$\Lambda_m = \{\lambda(v \to u)\}_{1 \leq u, v \leq N(m)}$ is the *transfer matrix* of the problem.

The total number of $m \times n$ arrangements is the sum of all of the entries of the $N(m) \times N(m)$ matrix Λ_m^{n-1}.

Hence the generating function for the number of m-rowed arrays, by their number of columns, is the sum of all of the entries of the matrix

$$xI + x^2\Lambda_m + x^3\Lambda_m^2 + \cdots = x(I - x\Lambda_m)^{-1}.$$

From linear algebra
$(B^{-1})_{ij} = (-1)^{i+j}\det(B : j, i)/\det B = \mathsf{cof}_{ij}(B)/\det B$. Thus,

$$x(I - x\Lambda_m)^{-1} = \frac{x\mathsf{cof}(I - x\Lambda_m)}{\det(I - x\Lambda_m)} = \frac{x\mathsf{cof}(I - x\Lambda_m)}{\prod_{i=1}^{N(m)}(1 - \lambda_i x)},$$

where cof is the $N(m) \times N(m)$ matrix of cofactors of $I - x\Lambda_m$ and the $\{\lambda_i\}$ are the eigenvalues of Λ_m.

If $m = 2$, $N(2) = (2+1)2^2 = 12$. Thus there are 12 possible pieces that might be used to make a $2 \times n$ array:

$$\begin{matrix} 4 \\ 4 \end{matrix}, \quad \begin{matrix} 1 \\ 4 \end{matrix}, \quad \begin{matrix} 1 \\ 1 \end{matrix}, \quad \begin{matrix} 3 \\ 4 \end{matrix}, \quad \begin{matrix} 4 \\ 3 \end{matrix}, \quad \begin{matrix} 2 \\ 4 \end{matrix}, \quad \begin{matrix} 1 \\ 3 \end{matrix}, \quad \begin{matrix} 1 \\ 2 \end{matrix}, \quad \begin{matrix} 2 \\ 1 \end{matrix}, \quad \begin{matrix} 2 \\ 2 \end{matrix}, \quad \begin{matrix} 2 \\ 3 \end{matrix}, \quad \begin{matrix} 3 \\ 3 \end{matrix}.$$

In this case, the sum of all of the entries of $x(I - x\Lambda_2)^{-1}$ is

$$\frac{12x - 29x^2 + 33x^3 - 9x^4}{(1-3x+x^2)(1-6x+9x^2)}.$$

The eigenvalues of the transfer matrix Λ_2 are 3 (twice), $(3 \pm \sqrt{5})/2$ (each twice), and 0 (six times).

The question is, in general what is the lagest eigenvalue of the $N(m) \times N(m)$ transfer matrix?

The 12 x 12 transfer matrix Λ_2 is

$$
\begin{array}{c|c|c|c|c|}
 & \binom{4}{4}\binom{1}{4}\binom{1}{1} & \binom{3}{4}\binom{2}{4}\binom{2}{1} & \binom{1}{3}\binom{1}{2}\binom{4}{3} & \binom{2}{2}\binom{2}{3}\binom{3}{3} \\
\hline
\begin{matrix}(44)\\(14)\\(11)\end{matrix} & 1 & 1 & 1 & 1 \\
\hline
\begin{matrix}(34)\\(12)\\(24)\end{matrix} & 0 & \begin{matrix}1 & 1 & 0\\1 & 1 & 1\\1 & 1 & 1\end{matrix} & 0 & 1 \\
\hline
\begin{matrix}(12)\\(13)\\(43)\end{matrix} & 0 & 0 & \begin{matrix}1 & 1 & 0\\1 & 1 & 1\\1 & 1 & 1\end{matrix} & 1 \\
\hline
\begin{matrix}(22)\\(23)\\(33)\end{matrix} & 0 & 0 & 0 & 1 \\
\hline
\end{array}
$$

In this figure, (ab) denotes $\binom{a}{b}$

Theorem. The largest eigenvalue of Λ_m is $(m+1)$, and it has multiplicity two.

From this theorem, using

$$x(I - x\Lambda_m)^{-1} = \frac{x\mathsf{cof}(I - x\Lambda_m)}{\prod_{i=1}^{N(m)}(1-\lambda_i x)}$$

and a partial fraction expansion for each entry of this matrix:

$$\frac{\alpha}{(1-x(m+1))^2} + \frac{\beta}{1-x(m+1)} + R(x),$$

where the partial fraction expansion of $R(x)$ involves only eigenvalues whose moduli are smaller than $m+1$, one deduces that

$$f_m(n) = (c_m n + d_m)(m+1)^n + O(\theta_m^n),$$

where $\theta_m < m+1$. Then by constructing explicitly $(n+1)(m+1)^n$ different nonattacking arrangements, we have that $c_m > 0$.

For $m = 2, 3$, asymptotically we have

$$f_2(n) = (17n - 109)3^n + O(2.618\ldots^n),$$

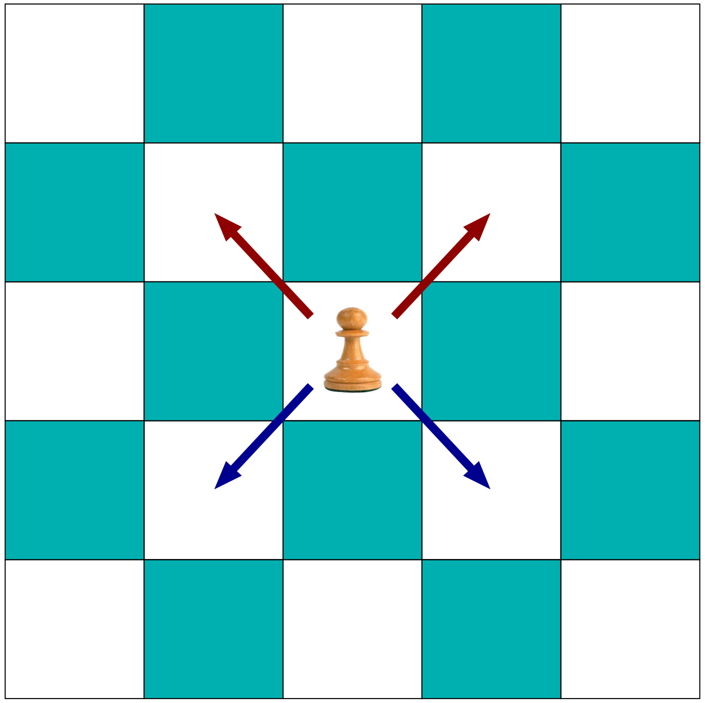
$$f_3(n) = (231n - 2377)4^n - 384 \cdot 3^n + O(1.707\ldots^n).$$

Also,

$$f_4(n) \sim \frac{7963567n}{2610} 5^n \sim 3051n \cdot 5^n.$$

S. Kitaev and T. Mansour: The problem of the pawns.

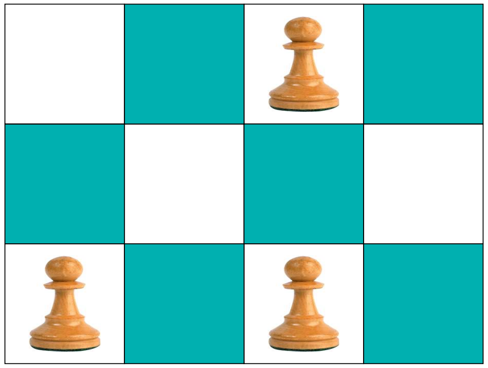
On an $m \times n$ chessboard, we place a number of nonattacking pawns, all of the same colour. *How many different placements are possible?*



The pawn can attack the cells pointed to by the red arrows.

However, we can assume that the pawn also can attack the cells pointed by the blue arrows, since pawns placed in these cells do attack the pawn in the figure, which makes these squares prohibited.

The placement of pawns is completely specified by an $m \times n$ binary matrix.



$\longleftrightarrow$

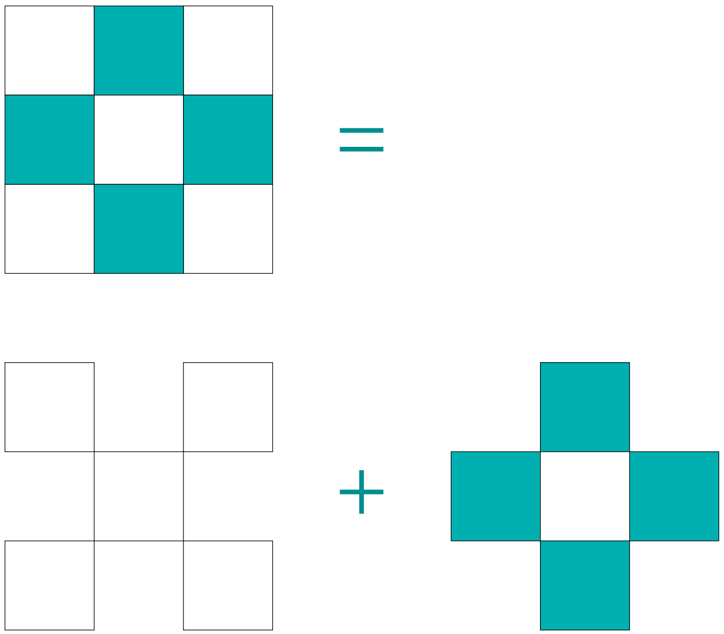
0	0	1	0
0	0	0	0
1	0	1	0

A legal matrix cannot contain the following two letter words:

$$\begin{array}{|c|c} \hline 1 & \multicolumn{1}{c}{} \\ \hline \multicolumn{1}{c|}{} & 1 \\ \cline{2-2} \end{array} \quad \text{and} \quad \begin{array}{c|c|} \cline{2-2} \multicolumn{1}{c}{} & 1 \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} .$$

$M_{m,n}$ – the number of the legal matrices

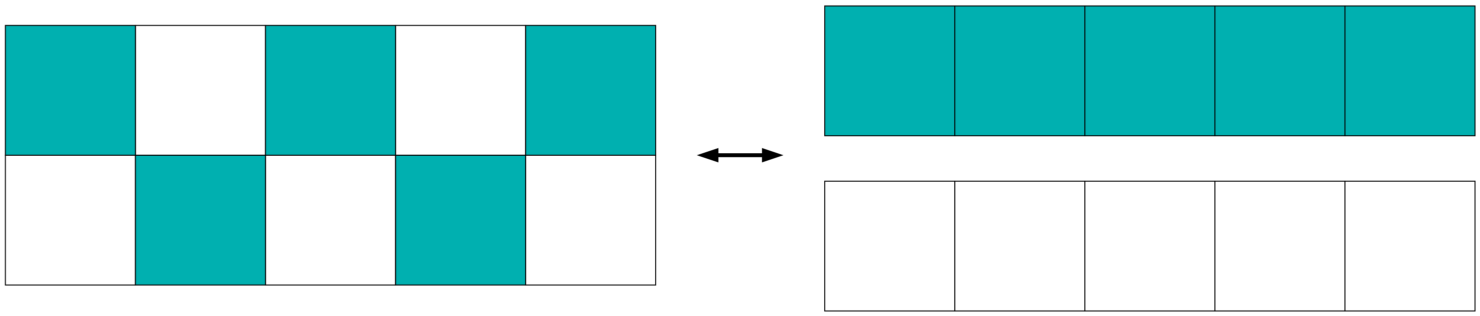
Observation. A pawn being placed on a square of a chessboard cannot attack a square of the other colour.



Theorem. We have that $M_{2m,n}$ is a perfect square.

Lemma. The number of binary strings of length n that avoid the word 11 is given by the Fibonacci number F_{n+1}.

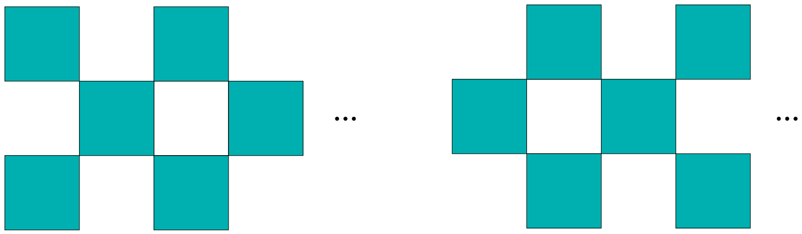
Proposition. We have that $M_{2,n} = (F_{n+1})^2$.



Proposition. For all $n \geq 0$, $M_{3,2n+1} = (4t_n - 3t_{n-1})(2t_n - 3t_{n-1})$ and $M_{3,2n} = t_n^2$, where

$$t_n = \frac{1}{\sqrt{13}} \left(\left(\frac{5+\sqrt{13}}{2} \right)^{n+1} - \left(\frac{5-\sqrt{13}}{2} \right)^{n+1} \right).$$

Proof.



If a_n and b_n denote the number of legal placements of pawns in the first and second shapes respectively, then $M_{3,n} = a_n b_n$,

$$a_n = 3a_{n-2} + b_{n-1}, \text{ and } b_n = a_{n-1} + b_{n-2}.$$

Proposition. We have that $M_{4,n} = \alpha_n^2$ and $M_{6,n} = \beta_n^2$, where the generating functions for the numbers α_n and β_n are given by

$$\frac{1 + 2x - 2x^2}{1 - 2x - 2x^2 + 2x^3}$$

and

$$\frac{1 + 5x - 9x^2 - 5x^3 + 6x^4}{1 - 3x - 6x^2 + 11x^3 + 5x^4 - 6x^5},$$

respectively.

Proposition. For all $n \geq 0$, $M_{5,n} = \alpha_n \beta_n$, where the generating functions for the numbers α_n and β_n are given by

$$\frac{1 + 7x - 4x^2 - 7x^3 + 5x^4}{(1+x)(1 - 2x - 6x^2 + 10x^3 - 4x^4)}$$

and

$$\frac{1 + 3x + x^2 - 5x^3 + 4x^4}{(1+x)(1 - 2x - 6x^2 + 10x^3 - 4x^4)},$$

respectively.

Upper bound for $M_{m,n}$

$U_{m,n}$ – the number of binary $m \times n$ matrices that avoid the word

$$\begin{array}{cc} \boxed{1} & \\ & \boxed{1} \end{array}$$

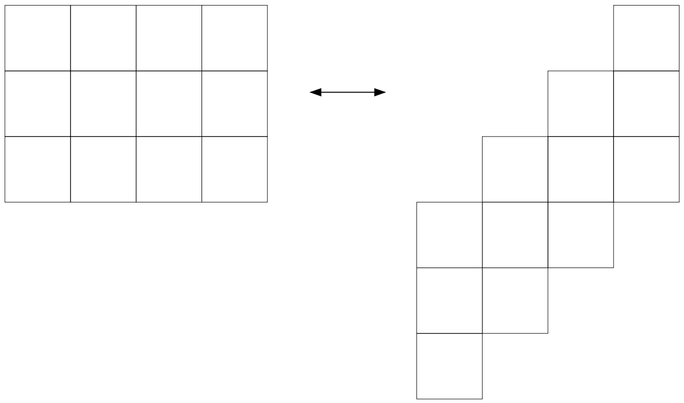
Clearly, $M_{m,n} \le U_{m,n}$.

Theorem. For any $n, m \ge 0$,

$$U_{m,n} = \begin{cases} F_{m+1}^{n-m+1} \left(\prod_{i=0}^{m} F_i \right)^2, & \text{if } n \ge m, \\ F_{n+1}^{m-n+1} \left(\prod_{i=0}^{n} F_i \right)^2, & \text{if } n < m, \end{cases}$$

where F_i is the i-th Fibonacci number.

The bijection that is used to prove the last theorem:



Asymptotics for the product of the first n Fibonacci numbers is

$$\frac{c}{\sqrt{5}^{n-1}}\left(\frac{1+\sqrt{5}}{2}\right)^{\frac{n(n-1)}{2}},$$

where $c = 1.226742010720353244417630 2\ldots$.

Theorem. We have

$$\lim_{n,m\to\infty}(M_{m,n})^{\frac{1}{mn}} \leq \frac{1+\sqrt{5}}{2}.$$

Lower bound for $M_{m,n}$

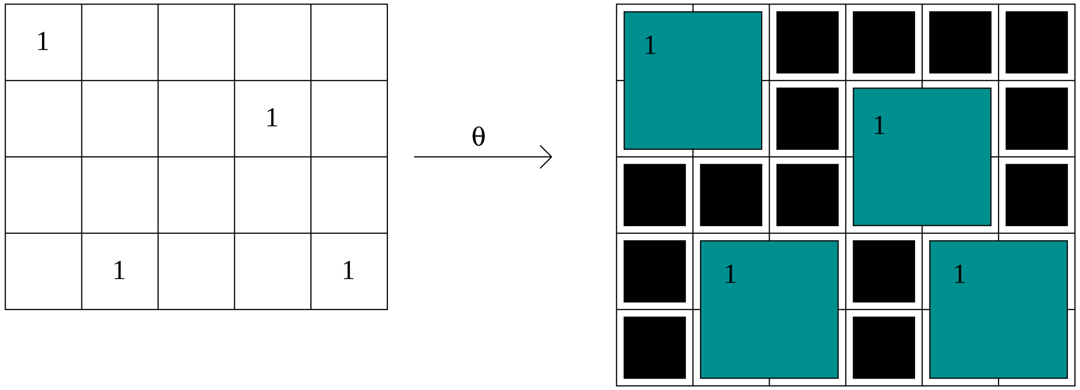
$L_{m,n}$ – the number of binary $m \times n$ matrices that avoid the words

$$\begin{array}{|c|}\hline 1 \\ \hline 1 \\ \hline \end{array}\,, \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array}\,, \quad \begin{array}{cc} \boxed{1} & \\ & \boxed{1} \end{array}\,, \text{ and } \begin{array}{cc} & \boxed{1} \\ \boxed{1} & \end{array}\,.$$

Clearly, $M_{m,n} \geq L_{m,n}$.

$L_{m,n}$ gives the number of *tilings of an* $(m+1) \times (n+1)$ *area with square tiles of size* 1×1 *and* 2×2.

The figure below shows the bijection θ between the problems in the case of a 4×5 matrix.



S. Heubach: Tiling an m-by-n area with squares of size up to k-by-k (m≤ 5).

S. Heubach and P. Chinn: Patterns arising from tiling rectangles with 1-by-1 and 2-by-2 squares.

Theorem. The number of tilings of a $3 \times n$ rectangle with squares of size 1×1 and 2×2 is given by $(2^{n+1} - (-1)^{n+1})/3$.

Theorem. The number of tilings of a $4 \times n$ rectangle with squares of size 1×1 and 2×2 is given by the recursive formula

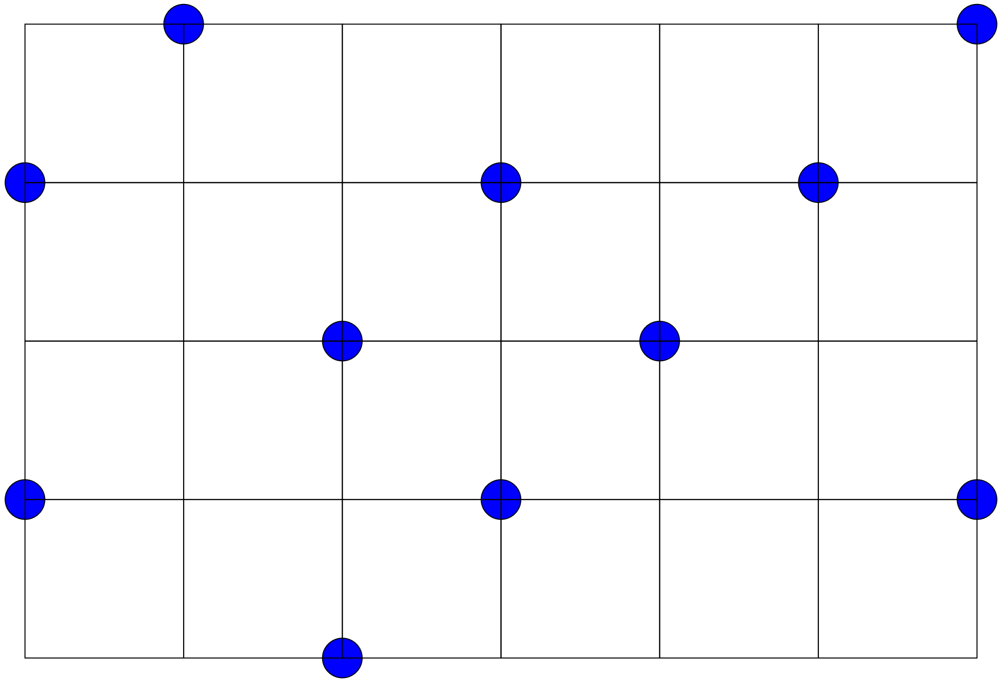
$$T_{4,n} = T_{4,n-1} + 4T_{4,n-2} + 2\sum_{k=0}^{n-3} T_{4,k},$$

with $T_{4,0} = T_{4,1} = 1$, and $T_{4,2} = 5$.

N. Calkin and H. Wilf:

The number of independent sets in a grid graph.

$G_{m,n}$ – the $m \times n$ *grid graph*.



Let $f(m,n)$ be the number of independent sets of vertices in $G_{m,n}$.

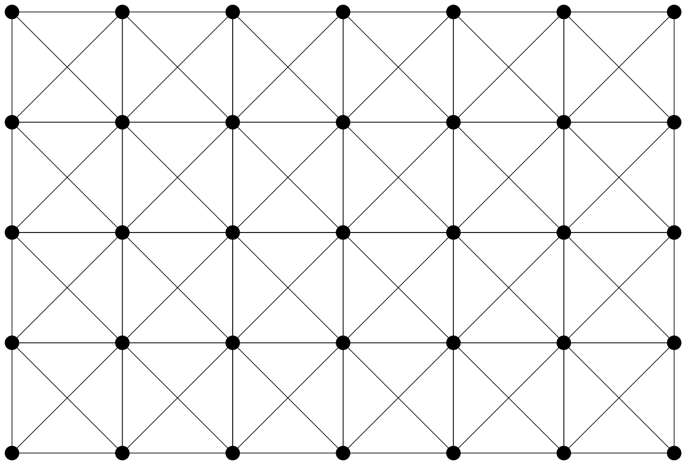
Clearly, $G_{m,n}$ contains an independent set of size $\geq mn/2$, and that set has $\geq 2^{mn/2}$ subsets, so certainly

$$\lim_{m,n\to\infty} f(m,n)^{\frac{1}{mn}} \geq 1.4142\ldots$$

The double limit $\eta = \lim_{m,n\to\infty} f(m,n)^{\frac{1}{mn}}$ exists and $1.503047782\ldots \leq \eta \leq 1.5035148\ldots$.

Numerical computations suggest that the true value of η is around $1.5030480824753323\ldots$...

The graph associated with kings:



The graph associated with pawns: