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Counting occurrences of 132 in a permutation

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Introduction

Let $\pi \in S_n$ and $\tau \in S_m$ be two permutations.

An *occurrence* of τ in π is a subsequence $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ such that $(\pi(i_1), \ldots, \pi(i_m))$ is order-isomorphic to τ; in such a context, τ is usually called a *pattern*.

Example: The permutation 315624 contains 132 exactly 6 times; the occurrences are 135, 136, 235, 236, 245, and 246.

Recently, much attention has been paid to the problem of counting the number $f_r^\tau(n)$ of permutations of length n containing a given number r of occurrences of a pattern τ.

Most authors consider only the case $r = 0$, thus studying permutations *avoiding* a given pattern. Only a few papers consider the case $r > 0$, usually restricting themselves to the patterns of length 3.

In fact, simple algebraic considerations show that there are only two different cases for $\tau \in S_3$, namely, $\tau = 123$ and $\tau = 132$.

Noonan (1996) has proved that

$$f_1^{123}(n) = \frac{3}{n}\binom{2n}{n-3}.$$

A general approach to the problem was suggested by Noonan and Zeilberger (1996); they gave another proof of Noonan's result, and conjectured that

$$f_2^{123}(n) = \frac{59n^2 + 117n + 100}{2n(2n-1)(n+5)}\binom{2n}{n-4}$$

and

$$f_1^{132}(n) = \binom{2n-3}{n-3}.$$

The first conjecture was proved by Fulmek (2001) and the latter by Bóna (1997).

A general conjecture of Noonan and Zeilberger (1996) states that $f_r^\tau(n)$ is P-recursive in n for any r and τ.

It was proved by Bóna for $\tau = 132$. However, a challenging question is to describe $f_r^\tau(n)$, $\tau \in S_3$, explicitly for any given r.

Here we suggest a new approach to this problem in the case $\tau = 132$, which allows to get an explicit expression for $f_r(n) = f_r^{132}(n)$ for any given r.

More precisely, we present an algorithm that computes the generating function $F_r(x) = \sum_{n\geq 0} f_r(n)x^n$ for any $r \geq 0$.

To get the result for a given r, the algorithm performs certain routine checks for each element of the symmetric group S_{2r}.

The algorithm has been implemented in C, and yielded explicit results for $1 \leq r \leq 6$.

Main result

First step.

To any $\pi \in S_n$ we assign a bipartite graph G_π in the following way.

The vertices in one part of G_π, denoted V_1, are the entries of π, and the vertices of the second part, denoted V_3, are the occurrences of 132 in π.

Entry $i \in V_1$ is connected by an edge to occurrence $j \in V_3$ if i enters j.

For example, let $\pi = 57614283$, then π contains 5 occurrences of 132 (576, 142, 143, 183, 283), and the graph G_π is presented in the following Figure:

Let $\widetilde{G}$ be an arbitrary connected component of G_π, and let $\widetilde{V}$ be its vertex set.

We denote $\widetilde{V}_1 = \widetilde{V} \cap V_1$, $\widetilde{V}_3 = \widetilde{V} \cap V_3$, $t_1 = |\widetilde{V}_1|$, $t_3 = |\widetilde{V}_3|$.

Lemma 1: For any connected component $\widetilde{G}$ of G_π one has $t_1 \leq 2t_3 + 1$.

Second step: Denote by G_π^n the connected component of G_π containing entry n.

Let $\pi(i_1), \ldots, \pi(i_s)$ be the entries of π belonging to G_π^n, and let $\sigma = \sigma_\pi \in S_s$ be the corresponding permutation.

We say that $\pi(i_1), \ldots, \pi(i_s)$ is the *kernel* of π and denote it $\ker \pi$;

σ is called the *shape* of the kernel, or the *kernel shape*,

s is called the *size* of the kernel,

and the number of occurrences of 132 in $\ker \pi$ is called the *capacity* of the kernel.

For example, for $\pi = 57614283$ as above, the kernel equals 14283, its shape is 14253, the size equals 5, and the capacity equals 4.

The following statement is implied immediately by Lemma 1.

Theorem 1: Let $\pi \in S_n$ contain exactly r occurrences of 132, then the size of the kernel of π is at most $2r+1$.

We say that ρ is a *kernel permutation* if it is the kernel shape for some permutation π.

Evidently ρ is a kernel permutation if and only if $\sigma_\rho = \rho$.

Let $\rho \in S_s$ be an arbitrary kernel permutation. We denote by $S(\rho)$ the set of all the permutations of all possible sizes whose kernel shape equals ρ.

Third step: For any $\pi \in S(\rho)$ we define the *kernel cell decomposition* as follows.

The number of cells in the decomposition equals $s(s+1)$.

Let $\ker \pi = \pi(i_1), \ldots, \pi(i_s)$; the *cell* $C_{m\ell} = C_{m\ell}(\pi)$ for $1 \le \ell \le s+1$ and $1 \le m \le s$ is defined by

$$\begin{aligned} C_{m\ell}(\pi) = \{\pi(j)\, & i_{\ell-1} < j < i_\ell, \\ & \pi(i_{\rho^{-1}(m-1)}) < \pi(j) < \pi(i_{\rho^{-1}(m)})\}, \end{aligned}$$

where $i_0 = 0$, $i_{s+1} = n+1$, and $\alpha(0) = 0$ for any α.

If π coincides with ρ itself, then all the cells in the decomposition are empty.

An arbitrary permutation in $S(\rho)$ is obtained by filling in some of the cells in the cell decomposition.

A cell C is called *infeasible* if the existence of an entry $a \in C$ would imply an occurrence of 132 that contains a and two other entries $x, y \in \ker\pi$. Clearly, all infeasible cells are empty for any $\pi \in S(\rho)$.

All the remaining cells are called *feasible*; a feasible cell may, or may not, be empty.

Consider the permutation $\pi = 67382451$. The kernel of π equals 3845, its shape is 1423.

The cell decomposition of π contains four feasible cells: $C_{13} = \{2\}$, $C_{14} = \varnothing$, $C_{15} = \{1\}$, and $C_{41} = \{6, 7\}$.

In the above example, all the other cells are infeasible; for example, C_{32} is infeasible, since if $a \in C_{32}$, then $a\pi'(i_2)\pi'(i_4)$ is an occurrence of 132 for any π' whose kernel is of shape 1423.

As another example,

$$\widetilde{\pi} = 10\,11\,7\,12\,4\,6\,5\,8\,3\,9\,2\,1$$

belongs to the same class $S(1423)$. Its kernel is $7\,12\,8\,9$, and the feasible cells are $C_{13} = \{4, 6, 5\}$, $C_{14} = \{3\}$, $C_{15} = \{2, 1\}$, $C_{41} = \{10, 11\}$.

Fourth step:

Given a cell C_{ij} in the kernel cell decomposition, all the kernel entries can be positioned with respect to C_{ij}.

We say that $x = \pi(i_k) \in \ker \pi$ lies *below* C_{ij} if $\rho(k) < i$, and *above* C_{ij} if $\rho(k) \geq i$.

Similarly, x lies to the *left* of C_{ij} if $k < j$, and to the *right* of C_{ij} if $k \geq j$.

As usual, we say that x lies to the *southwest* of C_{ij} if it lies below C_{ij} and to the left of it; the other three directions, northwest, southeast, and northeast, are defined similarly.

The following statement plays a crucial role in our considerations.

Lemma 3: Let $\pi \in S(\rho)$ and $\pi(i_k) \in \ker \pi$, then any cell $C_{m\ell}$ such that $\ell > k$ and $m > \rho(k)$ is infeasible.

As an easy corollary of Lemma 3, we get the following proposition. Let us define a partial order $\prec$ on the set of all feasible cells by saying that $C_{m\ell} \prec C_{m'\ell'} \neq C_{m\ell}$ if $m \geq m'$ and $\ell \leq \ell'$.

Lemma 4: $\prec$ is a linear order.

Consider now the dependence between two nonempty feasible cells lying on the same horizontal or vertical level.

Lemma 5: Let $C_{m\ell}$ and $C_{m\ell'}$ be two nonempty feasible cells such that $\ell < \ell'$. Then for any pair of entries $a \in C_{m\ell}$, $b \in C_{m\ell'}$, one has $a > b$.

Lemma 6: Let $C_{m\ell}$ and $C_{m'\ell}$ be two nonempty feasible cells such that $m < m'$. Then any entry $a \in C_{m\ell}$ lies to the right of any entry $b \in C_{m'\ell}$.

Lemmas 4–6 yield immediately the following two results.

Theorem 7 Let $\widetilde{G}$ be a connected component of G_π distinct from G_π^n. Then all the vertices in $\widetilde{V}_1$ belong to the same feasible cell in the kernel cell decomposition of π.

Let $F(\rho)$ be the set of all feasible cells in the kernel cell decomposition corresponding to permutations in $S(\rho)$, and let $f(\rho) = |F(\rho)|$.

We denote the cells in $F(\rho)$ by $C^1, \ldots, C^{f(\rho)}$ in such a way that $C^i \prec C^j$ whenever $i < j$.

Theorem 8: For any given sequence $\alpha_1, \ldots, \alpha_{f(\rho)}$ of arbitrary permutations there exists $\pi \in S(\rho)$ such that the content of C^i is order-isomorphic to α_i.

Fifth step: Let ρ be a kernel permutation, and let $s(\rho)$, $c(\rho)$, and $f(\rho)$ be the size of ρ, the capacity of ρ, and the number of feasible cells in the cell decomposition associated with ρ, respectively.

Denote by K the set of all kernel permutations, and by K_t the set of all kernel shapes for permutations in S_t.

Main Theorem: For any $r \geq 1$, the generating function $F_r(x)$ is given by

$$\sum_{\rho \in K_{2r+1}} \left(x^{s(\rho)} \sum_{r_1+\cdots+r_{f(\rho)}=r-c(\rho)} \prod_{j=1}^{f(\rho)} F_{r_j}(x) \right),$$

where $r_j \geq 0$ for $1 \leq j \leq f(\rho)$.

Main Theorem provides a finite algorithm for finding $F_r(x)$ for any given $r > 0$, since we have to consider all permutations in S_{2r+1}, and to perform certain routine operations with all shapes found so far. Moreover, the amount of search can be decreased substantially due to the following proposition.

PropositionL The only kernel permutation of capacity $r \geq 1$ and size $2r + 1$ is

$$2r-1, 2r+1, 2r-3, 2r, \dots,$$
$$2r-2j-3, 2r-2j, \dots 1, 4, 2.$$

Its contribution to $F_r(x)$ equals
$x^{2r+1}F_0^{r+2}(x)$.

By the above proposition, it suffices to search only permutations in S_{2r}. Below we present several explicit calculations.

Formula for $F_0(x)$:

Let us start from the case $r = 0$. Observe that Main Theorem remains valid for $r = 0$, provided the left hand side is replaced by $F_r(x) - 1$; subtracting 1 here accounts for the empty permutation. So, we begin with finding kernel shapes for all permutations in S_1. The only shape obtained is $\rho_1 = 1$, and it is easy to see that $s(\rho_1) = 1$, $c(\rho_1) = 0$, and $f(\rho_1) = 2$ (since both cells C_{11} and C_{12} are feasible). Therefore, we get

$$F_0(x) - 1 = xF_0^2(x),$$

which means that

$$F_0(x) = \frac{1 - \sqrt{1 - 4x}}{2x},$$

the generating function of Catalan numbers.

Formula for $F_1(x)$:

Let now $r = 1$. Since permutations in S_2 do not exhibit kernel shapes distinct from ρ_1, the only possible new shape is the exceptional one, $\rho_2 = 132$, whose contribution equals $x^3 F_0^3(x)$.

Therefore, Main Theorem amounts to

$$F_1(x) = 2xF_0(x)F_1(x) + x^3F_0^3(x),$$

and we get the following result (Bóna):

$$F_1(x) = \frac{1}{2}\Big(x - 1 + (1 - 3x)(1 - 4x)^{-1/2}\Big)\,;$$

equivalently, $f_1(n) = \binom{2n-3}{n-3}$ for $n \geq 3$.

Formula for $F_2(x)$:

Let $r = 2$. We have to check the kernel shapes of permutations in S_4. Exhaustive search adds four new shapes to the previous list; these are 1243, 1342, 1423, and 2143; besides, there is the exceptional $35142 \in S_5$.

Calculation of the parameters s, c, f is straightforward, and we get $F_2(x)$ is given by

$$\frac{1}{2}\Bigg(x^2 + 3x - 2 \\ +(2x^4 - 4x^3 + 29x^2 - 15x + 2)(1-4x)^{-3/2}\Bigg);$$

equivalently,

$$f_2(n) = \frac{n^3 + 17n^2 - 80n + 80}{2n(n-1)}\binom{2n-6}{n-2}$$

for $n \geq 4$.

Formula for $F_r(x)$ where $r = 3, 4, 5, 6$:

Let $r = 3, 4, 5, 6$; exhaustive search in S_6, S_8, S_{10}, and S_{12} reveals 20, 104, 503, and 2576 new nonexceptional kernel shapes, respectively, and we get for $3 \le r \le 6$

$$F_r(x) = \frac{1}{2}\left(P_r(x) + Q_r(x)(1-4x)^{-r+1/2}\right),$$

where

$$\begin{aligned}
P_3(x) &= 2x^3 - 5x^2 + 7x - 2, \\
P_4(x) &= 5x^4 - 7x^3 + 2x^2 + 8x - 3, \\
P_5(x) &= 14x^5 - 17x^4 + x^3 - 16x^2 + 14x - 2
\end{aligned}$$

and

$$\begin{aligned}
Q_3(x) &= -22x^6 - 106x^5 + 292x^4 - 302x^3 \\
&\quad +135x^2 - 27x + 2, \\
Q_4(x) &= 2x^9 + 218x^8 + 1074x^7 - 1754x^6 \\
&\quad +388x^5 + 1087x^4 - 945x^3 + 320x^2 \\
&\quad -50x + 3, \\
Q_5(x) &= -50x^{11} - 2568x^{10} - 10826x^9 \\
&\quad +16252x^8 - 12466x^7 + 16184x^6 \\
&\quad -16480x^5 + 9191x^4 - 2893x^3 + 520x^2 \\
&\quad -50x + 2.
\end{aligned}$$

Equivalently,

$$f_r(n) = R_r(n)\frac{(2n-3r)!}{n!r!(n-r-2)!},$$

for $n \geq r+2$, where

$$\begin{aligned}
R_3(n) \;&= n^6 + 51n^5 - 407n^4 - 99n^3 + 7750n^2 \\
&\quad -22416n + 20160, \\[2ex]
R_4(n) \;&= n^9 + 102n^8 - 282n^7 - 12264n^6 \\
&\quad +32589n^5 + 891978n^4 - 7589428n^3 \\
&\quad +25452024n^2 - 39821760n + 23950080, \\[2ex]
R_5(n) \;&= n^{12} + 170n^{11} + 1861n^{10} \\
&\quad -88090n^9 - 307617n^8 + 27882510n^7 \\
&\quad -348117457n^6 + 2119611370n^5 \\
&\quad -6970280884n^4 + 10530947320n^3 \\
&\quad +2614396896n^2 - 30327454080n \\
&\quad +29059430400.
\end{aligned}$$

The expressions for $P_6(x)$, $Q_6(x)$, and $R_6(n)$ are too long to be presented here.

Further results and open questions:

As an easy consequence of Theorem 4 we get the following results due to Bóna.

Corollary 1: Let $r \geq 0$, then $F_r(x)$ is a rational function in the variables x and $\sqrt{1-4x}$.

In fact, Bóna has proved a stronger result, claiming that

$$F_r(x) = P_r(x) + Q_r(x)(1-4x)^{-r+1/2}, \quad (*)$$

where $P_r(x)$ and $Q_r(x)$ are polynomials and $1-4x$ does not divide $Q_r(x)$. We were unable to prove this result; however, it stems almost immediately from the following conjecture.

Conjecture 1: For any kernel permutation $\rho \neq 1$, $s(\rho) \geq f(\rho)$.

In view of our explicit results, we have even a stronger conjecture.

Conjecture 2: The polynomials $P_r(x)$ and $Q_r(x)$ in $(*)$ have halfinteger coefficients.

Another direction would be to match the approach of this note with the previous results on restricted 132-avoiding permutations.

Let $\Phi_r(x;k)$ be the generating function for the number of permutations in S_n containing r occurrences of 132 and avoiding the pattern $12\ldots k \in S_k$.

It was shown previously that $\Phi_r(x;k)$ can be expressed via Chebyshev polynomials of the second kind for $r = 0$ (Chow and West) and $r = 1$ (Mansour and Vainshtein). Our new approach allows to get a recursion for $\Phi_r(x;k)$ for any given $r \geq 0$.

Let ρ be a kernel permutation, and assume that the feasible cells of the kernel cell decomposition associated with ρ are ordered linearly according to $\prec$. We denote by $l_j(\rho)$ the length of the longest increasing subsequence of ρ that lies to the north-east from C^j.

For example, let $\rho = 1423$, as on Figure 2.

Then $\ell_1(\rho) = 1$, $\ell_2(\rho) = 2$, $\ell_3(\rho) = 1$, $\ell_4(\rho) = 0$.

Corollary 2: For any $r \geq 1$ and $k \geq 3$, the generating function $\Phi_r(x;k)$ is given by

$$\sum_{\rho \in K_{2r+1}} \left(x^{s(\rho)} \sum \prod_{j=1}^{f(\rho)} \Phi_{r_j}(x; k - \ell_j(\rho)) \right),$$

where $\Phi_r(x;m) \equiv 0$ for $m \leq 0$, and the inner sum over all $r_j \geq 0$ for $1 \leq j \leq f(\rho)$ such that $r_1 + \cdots + r_{f(\rho)} = r - c(\rho)$.

As in the case of $F_r(x)$, the statement of the theorem remains valid for $r = 0$, provided the left hand side is replaced by $\Phi_r(x;k) - 1$.

This allows to recover known explicit expressions for $\Phi_r(x;k)$ for $r = 0,1$, and to get an expression for $r = 2$, which is too long to be presented here.

This approach can be extended even further, to cover also permutations containing r occurrences of 132 and avoiding other permutations in S_k, for example, $23\dots k1$.