Restricted permutations and Chebyshev polynomials

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Introduction

Let $\pi \in S_n$ and $\tau \in S_k$ be two permutations. An *occurrence* of τ in π is a subsequence

$$1 \leq i_1 < i_2 < \cdots < i_k \leq n$$

such that $(\pi_{i_1}, \ldots, \pi_{i_k})$ is order isomorphic to τ; in such a context τ is usually called a *pattern*. We say that π *avoids* τ, or is τ-*avoiding*, if there is no occurrence of τ in π. The set of all τ-avoiding permutations in S_n is denoted $S_n(\tau)$.

Example 1:

$S_n(12) = \{(n, n-1, \ldots, 1)\}$

and

$S_n(21) = \{(1, 2, \ldots, n)\}$.

Example 2:

The first case to be examined was the case of permutations avoiding one pattern of length 3. Knuth found that

$$|S_n(\tau)| = c_n = \frac{1}{n+1}\binom{2n}{n}$$

for any $\tau \in S_3$.

For an arbitrary finite collection of patterns T, we say that π avoids T if π avoids any $\tau \in T$; the corresponding subset of S_n is denoted $S_n(T)$. By F_τ and F_T we denote the corresponding (ordinary) generating functions

$$F_\tau(x) = \sum_{n=0}^{\infty} |S_n(\tau)| x^n$$

and

$$F_T(x) = \sum_{n=0}^{\infty} |S_n(T)| x^n.$$

The first paper devoted entirely to the study of permutations avoiding certain patterns (*restricted permutations*) appeared in 1985 (see Simion and Schmidt [SS]). Currently there exists more than hundred papers on this subject.

Chebyshev polynomials of the second kind (in what follows just Chebyshev polynomials) are defined by

$$U_r(\cos\theta) = \frac{\sin(r+1)\theta}{\sin\theta}$$

for $r \geq 0$. Evidently, $U_r(x)$ is a polynomial of degree r in x with integer coefficients.

Apparently, for the first time the relation between restricted permutations and Chebyshev polynomials was discovered by Chow and West in [CW]. The main result of [CW] can be formulated as follows.

For $k \geq 1$ we define $R_k(x)$ by

$$R_k(x) = \frac{2tU_{k-1}(t)}{U_k(t)}, \qquad t = \frac{1}{2\sqrt{x}}.$$

It is easy to see that $R_k(x)$ is rational in x.

Theorem (Chow and West [CW]).

Let $T_1 = \{321, (2, 3, \ldots, k, 1)\}$, $T_2 = \{132, (1, 2, \ldots, k)\}$, and $T_3 = \{132, (2, 3, \ldots, k, 1)\}$, then

(i) $F_{T_1}(x) = R_k(x)$;

(ii) $F_{T_2}(x) = R_k(x)$;

(iii) $F_{T_3}(x) = R_k(x)$.

The original proof is based on the use of transfer matrices. Several different proofs and different extensions of various parts of this theorem appeared recently.

In 1999, Robertson [R] proved the following.

Theorem (Robertson, [R]).

The number of permutations containing exactly once 132 which is 123-avoiding, is given by

$$(n-2)2^{n-3}$$

for $n \geq 3$. Also, the number of permutations containing exactly one 123 and one 132 equals

$$(n-3)(n-4)2^{n-5}.$$

Later, Robertson, Wilf and Zeilbereger [RWZ] gave the generating function for the number of 132-avoiding permutations that contain a prescribed number of 123.

Theorem (Robertson, Wilf, and Zeilberger [RWZ]).

The generating function for the number of 132-avoiding permutations that contain a prescribed number of 123 is given by the continued fraction

$$\cfrac{1}{1-\cfrac{xy^{d_1}}{1-\cfrac{xy^{d_2}}{1-\cfrac{xy^{d_3}}{1-\ddots}}}},$$

where $d_i = \binom{i-1}{2}$, and $\binom{a}{b}$ is assumed 0 whenever $a < b$ or $b < 0$.

Mansour and Vainshtein gave a generalization of this theorem, by use the same argument proof as follows.

Let $f_n^r(k)$ stand for the number of 132-avoiding permutations on n letters that contain exactly r occurrences of $12\ldots k$. We denote by $F(x,y;k)$ the generating function of the sequence $\{f_n^r(k\})$, that is,

$$F(x,y;k) = \sum_{n\geq 0}\sum_{r\geq 0} f_n^r(k)x^ny^r.$$

Our first result is a natural generalization of the main theorem of [RWZ].

Theorem (Mansour and Vainshtein [MV1]).

The generating function $F(x,y;k)$ for $k \geq 1$ is given by the continued fraction

$$F(x,y;k) = \cfrac{1}{1 - \cfrac{xy^{d_1}}{1 - \cfrac{xy^{d_2}}{1 - \cfrac{xy^{d_3}}{\cdots}}}},$$

where $d_i = \binom{i-1}{k-1}$, and $\binom{a}{b}$ is assumed 0 whenever $a < b$ or $b < 0$.

By the above theorem, Mansour and vainshtein gave an extension for Chow and West Theorem.

Theorem (Mansour and Vainshtein [MV1]).

For any $k \geq 1$, $F_r(x;k)$ is a rational function given by

$$F_r(x;k) = \frac{x^{\frac{r-1}{2}} U_{k-1}^{r-1}\left(\frac{1}{2\sqrt{x}}\right)}{U_k^{r+1}\left(\frac{1}{2\sqrt{x}}\right)}, \quad 1 \leq r \leq k,$$

$$F_0(x;k) = \frac{U_{k-1}\left(\frac{1}{2\sqrt{x}}\right)}{\sqrt{x}U_k\left(\frac{1}{2\sqrt{x}}\right)}.$$

Theorem (Mansour and Vainshtein [MV1]).

For any $k \geq 1$ and $1 \leq r \leq k(k+3)/2$, $F_r(x;k)$ is a rational function given by

$$\frac{x^{\frac{r-1}{2}} U_{k-1}^{r-1}\left(\frac{1}{2\sqrt{x}}\right)}{U_k^{r+1}\left(\frac{1}{2\sqrt{x}}\right)} \sum_{j=0}^{\lfloor (r-1)/k \rfloor} \binom{r-kj+j-1}{j} \left(\frac{U_k\left(\frac{1}{2\sqrt{x}}\right)}{x^{\frac{k-2}{2k}} U_{k-1}\left(\frac{1}{2\sqrt{x}}\right)} \right)^{kj}.$$

Later, Krattenthaler [Kr] exhibit a bijection between 132-avoiding permutations and Dyck paths, and using this bijection to discover results on generating functions for 132-avoiding permutations with a given number of occurrences of the pattern $12...k$. Moreover, he exhibited a bijection between 123-avoiding permutations and Dyck paths which allows to express the generating function for 123-avoiding permutations with a given number of occurrences of the pattern $(k-1,\ldots,1,k)$ in form of a continued fraction.

Theorem (Krattenthaler, [Kr]).

(i) Let $N_k(\pi)$ be the number of occurrences of the pattern $12...k$ in π, and the length of π denoted by $|\pi|$. Then the generating function $\sum_\pi y^{N_k(\pi)} x^{|\pi|}$, where the sum is over all 132-avoiding permutations, is given by

$$\cfrac{1}{1-\cfrac{xy^{d_1}}{1-\cfrac{xy^{d_2}}{1-\cfrac{xy^{d_3}}{1-\ddots}}}},$$

where $d_i = \binom{i-1}{k-1}$, and $\binom{a}{b}$ is assumed 0 whenever $a < b$ or $b < 0$;

(ii) The generating function $\sum_\pi x^{|\pi|}$, where the sum is over all 132-avoiding permutations with exactly r occurrences of the pattern $12...k$, is given by

$$\sum \left(\binom{l_1+l_2-1}{l_2} \binom{l_2+l_3-1}{l_3} \cdots \right) \frac{U_k^{l_1-1}(1/(2\sqrt{x}))}{U_k^{l_1+1}(1/(2\sqrt{x}))} x^{\frac{1}{2}(l_1-1)+l_2+l_3+\dots},$$

where the sum is over all nonnegative integers $l_1, l_2, \dots$ with

$$l_1 \binom{k-1}{k-1} + l_2 \binom{k}{k-1} + l_3 \binom{k+1}{k-1} + \cdots = r;$$

(iii) The generating function $\sum_\pi x^{|\pi|}$, where the sum is over all 132-avoiding permutations which also avoids the pattern $23 \dots k1$, is given by $R_k(x)$;

(iv) The generating function $\sum_\pi x^{|\pi|}$, where the sum is over all 123-avoiding permutations which also avoids the pattern $(k-1)(k-2) \dots 1k$, is given by $R_k(x)$;

(v) The generating function $sf \sum_\pi x^{|\pi|}$, where the sum is over all 132-avoiding permutations with exactly one occurrence of the pattern $23 \dots k1$, is given by $\frac{x}{U_{k-2}(t)U_k(t)}$;

(vi) The generating function $\sum_\pi x^{|\pi|}$, where the sum is over all 123-avoiding permutations with exactly r, $1 \le r \le k-1$ occurrences of the pattern $23...k1$, is given by $x^{(r-1)/2}\frac{U_{k-1}^{r-1}(t)}{U_k^{r+1}(t)}$;

where $t = \frac{1}{2\sqrt{x}}$.

Now, we will describe several ways to extend and generalize the above result. All our results deal with multiple (mainly, double) restrictions, of which one belongs to S_3 and others to S_k. Some of the results have been published previously, and we state them here without a proof. Observe that modulo standard symmetry operations (complement, reversal, and inversion), there are only two nonequivalent patterns in S_3; we choose them to be 132 and 321.

For the sake of brevity, we denote by $[k]$ the identity pattern $(1, 2, \ldots, k)$, and by $[k, m]$ the two-layered pattern

$$(m+1, m+2, \ldots, k, 1, 2, \ldots, m).$$

In general, we say that $\tau \in S_k$ is a layered pattern if it can be represented as $\tau = (\tau^0, \tau^1, \ldots, \tau^r)$, where each of τ^i is a nonempty permutation of the form

$$\tau^i = (m_{i+1}+1, m_{i+1}+2, \ldots, m_i)$$

with $k = m_0 > m_1 > \cdots > m_r > m_{r+1} = 0$; in this case we denote τ by $[m_0, \ldots, m_r]$.

Now, the question is:

What directions there exist to extend and generalize the above result ?????

132-restricted permutations and 321-restricted permutations

(1) Avoiding 132 and another pattern;

(2) Avoiding 132 and containing another pattern;

(3) Containing 132 exactly once and avoiding another pattern;

(4) Containing 132 and another pattern exactly once;

(5) Generalizations;

(6) 321-restricted permutations.

(7) Other directions......

(1) Avoiding 132 and another pattern

Denote by $\bar{F}_T(x) = F_{132,T}(x)$ the generating function for the number of permutations in $S_n(132, T)$.

By Theorem Chow and West parts (ii) and (iii), the pairs $\{132, [k]\}$ and $\{132, [k, 1]\}$ are Wilf-equivalent, that is $\bar{F}_{[k]}(x) = \bar{F}_{[k,1]}(x)$. It turns out that this equivalence class can be extended as follows.

Theorem 1 ([MV2, Th. 2.4]). For any m, $1 \leq m \leq k-1$,

$$\bar{F}_{[k,m]}(x) = R_k(x).$$

A further extension is provided by the following result. We say that $\tau \in S_k$ is a wedge pattern if it can be represented as $\tau = (\tau^1, \rho^1, \dots, \tau^r, \rho^r)$ so that each of τ^i is nonempty, $(\rho^1, \rho^2, \dots, \rho^r)$ is a layered permutation of $1, \dots, s$ for some s, and $(\tau^1, \tau^2, \dots, \tau^r) = (s+1, s+2, \dots, k)$. For example, 645783912 is a wedge pattern; here $r = 3$, $s = 5$, $\tau^1 = (6)$, $\tau^2 = (7, 8)$, $\tau^3 = (9)$, $\rho^1 = (4, 5)$, $\rho^2 = (3)$, $\rho^3 = (1, 2)$. Evidently, $[k, m]$ is a wedge pattern for any m.

Theorem 2 ([MV2, Th. 2.6]). $\bar{F}_\tau(x) = R_k(x)$ for any wedge pattern $\tau \in S_k(132)$.

The proofs of the above two results are purely analytical. It would be very interesting to find a bijective proof of these results.

The case of general layered patterns can be also expressed in terms of Chebyshev polynomials. Since the expressions become rather cumbersome, we present here only the case of a 3-layered pattern.

Theorem 3 ([MV2, Th. 2.5]). For any $k > m_1 > m_2 > 0$,

$$\bar{F}_{[k,m_1,m_2]}(x) = 2t\frac{U_{\alpha+\beta}(t)U_{\alpha+\gamma-1}(t)U_{\beta+\gamma}(t) + U_{\beta-1}(t)U_{\beta}(t)}{U_{\alpha+\beta}(t)U_{\alpha+\gamma}(t)U_{\beta+\gamma}(t)},$$

where $\alpha = k - m_1$, $\beta = m_1 - m_2$, $\gamma = m_2$, $t = \frac{1}{2\sqrt{x}}$.

(2) Avoiding 132 and containing another pattern

Denote by $G_\tau^r(x)$ the generating function for the number of permutations in $S_n(132)$ containing a pattern $\tau \in S_k$ exactly r times. We start from the following result obtained by Mansour and Vainshtein [MV1].

Theorem 4 ([MV1, Th 3.1]). For any $k \geq 1$,

$$G_{[k]}^1(x) = \frac{1}{U_k^2(t)}, \qquad t = \frac{1}{2\sqrt{x}}.$$

This result may be extended in two directions. First, let us fix $r = 1$ and consider other patterns τ. The case of $\tau = [k, 1]$ was studied by Krattenthaler [Kr].

Theorem 5 ([Kr, Th. 7]). For $k \geq 2$,

$$G_{[k,1]}^1(x) = \frac{1}{4t^2 U_{k-2}(t) U_k(t)}, \qquad t = \frac{1}{2\sqrt{x}}.$$

A more general case of $\tau = [k, m]$ is investigated by Mansour and Vainshtein [MV2].

Theorem 6 ([MV2, Th. 3.4]). For any $k > m > 0$,

$$G^1_{[k,m]}(x) = \frac{1}{2tU_k(t)U_m(t)U_{k-m-1}(t)}, \qquad t = \frac{1}{2\sqrt{x}}.$$

Another direction would be to increase the value of r. A generalization Theorem 5 to the case $1 \leq r \leq k$ was obtained by Mansour and Vainshtein [MV1].

Theorem 7 ([MV1, Th. 3.1]). For any r, $1 \leq r \leq k$,

$$G^r_{[k]}(x) = \frac{U^{r-1}_{k-1}(t)}{(2t)^{r-1}U^{r+1}_k(t)}, \qquad t = \frac{1}{2\sqrt{x}}.$$

A similar generalization of Theorem 5 is as follows.

Theorem 8 ([Kr, Th. 7]). For any r, $1 \leq r \leq k-1$,

$$G^r_{[k,1]}(x) = \frac{1}{U_{k-3}(t)U_k(t)} \sum_{l|r} \frac{1}{l+1}\binom{2l}{l}(2t)^{1-2l-\frac{r}{l}}\left(\frac{U_{k-3}(t)}{U_{k-2}(t)}\right)^{r/l},$$

where $t = \frac{1}{2\sqrt{x}}$.

Theorem 7 can be extended to cover a wider range of r's.

Theorem 9 ([MV1, Th. 4.1]). For any r, $1 \leq r \leq k(k+3)/2$,

$$G^r_{[k]}(x) = \frac{U^{r-1}_{k-1}(t)}{(2t)^{r-1}U^{r+1}_k(t)} \sum_{j=0}^{\lfloor (r-1)/k \rfloor} \binom{r-kj+j-1}{j} \left(\frac{(2t)^{\frac{k-2}{k}} U_k(t)}{U_{k-1}(t)} \right)^{kj},$$

where $t = 1/2\sqrt{x}$.

The case of general r was treated by Krattenthaler [Kr].

Theorem 10 ([Kr, Th. 3]). For any r,

$$G^r_{[k]}(x) = \sum \left(\binom{l_1+l_2-1}{l_2} \binom{l_2+l_3-1}{l_3} \dots \right) \cdot \cdot \frac{U^{l_1-1}_{k-1}(t)}{U^{l_1+1}_k(t)} (2t)^{-(l_1-1)-2(l_2+l_3+\dots)},$$

where the sum is over all nonnegative integers $l_1, l_2, \dots$ with

$$l_1 \binom{k-1}{k-1} + l_2 \binom{k}{k-1} + l_3 \binom{k+1}{k-1} + \cdots = r,$$

and $t = 1/2\sqrt{x}$.

(3) Containing 132 exactly once and avoiding another pattern

Denote by $H_\tau(x)$ the generating function for the number of permutations in S_n avoiding a pattern $\tau \in S_k$ and containing 132 exactly once. We start from the following result obtained by Mansour and Vainshtein [MV1].

Theorem 11 ([MV1, Th 4.2]). For any $k \geq 3$,

$$H_{[k]}(x) = \frac{1}{4t^2 U_k^2(t)} \sum_{j=1}^{k-2} U_j^2(t), \qquad t = \frac{1}{2\sqrt{x}}.$$

This result can be extended to the case of general 2-layered patterns as follows.

Theorem 12 (MV). (i) For any $k \geq 4$,

$$H_{[k,1]}(x) = \frac{1}{4t^2U_k^2(t)} \left(\sum_{j=1}^{k-2} U_j^2(t) - 1 \right), \quad t = \frac{1}{2\sqrt{x}};$$

besides,

$$H_{[3,1]}(x) = \frac{x^3}{1-2x}.$$

(ii) For any $k \geq 5$,

$$H_{[k,2]}(x) = \frac{1}{4t^2U_k^2(t)} \left(\sum_{j=1}^{k-2} U_j^2(t) - \frac{2tU_{k-3}(t)}{U_{k-2}(t)} - 2 \right), \quad t = \frac{1}{2\sqrt{x}};$$

besides,

$$H_{[4,2]}(x) = \frac{x^3(1+x)}{(1-x)(1-3x+x^2)}.$$

(iii) For any $k \geq 6$ and any m, $3 \leq m \leq k/2$,

$$H_{[k,m]}(x) = \tfrac{1}{4t^2U_k^2(t)} \Big(\textstyle\sum_{j=1}^{k-m-2} U_j^2(t) + \sum_{j=1}^{m-1} U_j^2(t) - 1$$
$$+U_{k-1}(t)U_{m-1}(t)U_{k-m-2}(t) \Big), \quad t = \tfrac{1}{2\sqrt{x}}.$$

Remark: Clearly, the cases $k = 3$, $m = 2$; $k = 4, 5$, $m = 3$; $k \geq 6$, $m > k/2$ are also covered, since $H_{[k,m]}(x) = H_{[k,k-m]}(x)$.

(4) Containing 132 and another pattern exactly once

Denote by $\Phi_\tau(x)$ the generating function for the number of permutations in S_n containing both 132 and a pattern $\tau \in S_k$ exactly once. We start from the following result.

Theorem 13 (MV). For any $k \geq 1$,

$$\Phi_{[k]}(x) = \frac{1}{4t^3 U_k^2(t)} \sum_{i=1}^{k-2} \frac{\sum_{j=1}^{k-i} U_j^2(t) - 1}{U_{k-i}(t)U_{k-i+1}(t)}, \qquad t = \frac{1}{2\sqrt{x}}.$$

In particular, for $k = 3$ we get $\Phi_{[3]}(x) = 2x^5(1-2x)^{-3}$, which means that the number of permutations in S_n containing both 132 and 123 exactly once equals $(n-3)(n-4)2^{n-5}$ (see Robertson Theorem).

Similarly to the previous section, this result can be extended to the case of general 2-layered patterns. Since the answers become very cumbersome, we present here only the simplest case.

Theorem 14 (MV). For any $k \geq 4$,

$$\Phi_{[k,1]}(x) = \frac{1}{8t^3 U_k^2(t)} \left(\frac{U_k(t)}{2t^2 U_{k-2}(t)} \sum_{i=1}^{k-4} \frac{\sum_{j=1}^{k-i-2} U_j^2(t) - 1}{U_{k-i-2}(t) U_{k-i-1}(t)} \right.$$
$$\left. + \frac{1}{2t U_{k-2}^2(t)} \sum_{i=1}^{k-4} U_i^2(t) + \frac{U_{k-3}(t)}{2t U_{k-1}(t)} + \frac{U_k(t)}{U_{k-1}(t)} \right),$$

where $t = 1/2\sqrt{x}$.

(5) Generalizations

Here we present several directions to generalize the results of the previous sections. First of these directions is to consider more than one additional restriction. For example, the following result is true. Let $\bar{F}_{\tau_1,\tau_2}(x)$ be the generating function for the number of permutations in $S_n(132,\tau_1,\tau_2)$. Assume that $\tau_1 = [k,m]$, $k-m \geq m$, and $\tau_2 = [l]$. It is easy to see that the only interesting case is $l > k-m$, since otherwise $\bar{F}_{[k,m],[l]}(x) = \bar{F}_{[l]}(x)$.

Theorem 15 (MV). Let $l > k-m \geq m$, then

$$\bar{F}_{[k,m],[l]}(x) = R_k(x) - \big(xR_{k-m}(x)R_m(x)\big)^{l+m-k}(R_k(x) - R_{k-m}(x)).$$

Similarly, let $G_{\tau_1;\tau_2}(x)$ be the generating function for the number of permutations in $S_n(132,\tau_1)$ that contain τ_2 exactly once. As before, we assume $\tau_1 = [k,m]$, $k-m \geq m$, and $\tau_2 = [l]$. Once again, the case $m \geq l$ is of no interest, since in this case $G_{[k,m];[l]}(x) = G_{[l]}(x)$.

Theorem 16 (MV). (i) Let $l > k-m \geq m$, then

$$G_{[k,m];[l]}(x) = \tfrac{1}{U_m(t)U_{k-m}(t)} \left(\tfrac{U_{m-1}(t)}{U_m(t)}\right)^{l-m} \left(\tfrac{U_{k-m-1}(t)}{U_{k-m}(t)}\right)^{l+m-k}$$
$$\times \left(\textstyle\sum_{j=m+2}^{k-m} \tfrac{U_{j-m-2}(t)}{U_{j-2}(t)U_{j-1}(t)} \left(\tfrac{U_{m-1}(t)}{U_m(t)}\right)^{m+1-j} + 1\right),$$

where $t = 1/2\sqrt{x}$.

(ii) Let $k - m \geq l > m$, then

$$G_{[k,m];[l]}(x) = \frac{1}{U_l(t)U_m(t)}\left(\frac{U_{m-1}(t)}{U_m(t)}\right)^{l-m}\left(\sum_{j=m+1}^{l}\frac{U_{j-m-1}(t)}{U_{j-1}(t)U_j(t)}\left(\frac{U_{m-1}(t)}{U_m(t)}\right)^{m-j}+1\right),$$

where $t = 1/2\sqrt{x}$.

Another possible direction is to replace 132 by some restriction of length 4 or more having a similar restrictive power. Define L_p as the set of all patterns in S_p of the form $\pi_1 1 \pi_2 2 \pi_3$, where π_2 is nonempty. Evidently, $L_3 = \{132\}$; for $p = 4$ we get $L_4 = \{1324, 1423, 1342, 1432, 3142, 4132\}$, and so on. It turns out that L_p is, in a sense, an analog of 132 for $p > 3$. For example, the following result is an analog of Chow an West Theorem (ii) for $p = 4$.

Theorem 17 (MV). For any $k \geq 2$,

$$F_{\{L_4,[k]\}}(x) = 1 + x + x^2 R_k(x) R_{k-1}(x)\big(R_{k-1}(x) + R_{k-2}(x)\big).$$

Consider now the case of a general $p > 3$. For an arbitrary $\pi \in S_p$ we define a sequence $a(\pi)$ of zeros and ones of length $p-1$ as follows. First of all, we put $a_1(\pi) = 1$ if $\pi_{p-1} < \pi_p$ and $a_1(\pi) = 0$ otherwise. If $a_1(\pi), \dots, a_j(\pi)$ are already determined, we put $a_{j+1}(\pi) = 1$ if the length of the maximal increasing subsequence in $(\pi_{k-j-1}, \pi_{k-j}, \dots, \pi_k)$ is greater than that of $(\pi_{k-j}, \dots, \pi_k)$, and $a_{j+1}(\pi) = 0$ otherwise. For example, let $\pi = 7346215$, then $a_1(\pi) = 1$, $a_2(\pi) = 0$, $a_3(\pi) = 0$, $a_4(\pi) = 0$, $a_5(\pi) = 1$, $a_6(\pi) = 0$. For an arbitrary sequence $a = (a_1, \dots, a_{p-1}) \in Q^{p-1} = \{0, 1\}^{p-1}$ we define by $N(a)$ the number of permutations $\pi \in S_p$ such that $a(\pi) = a$. The following result is a further generalization of Theorem 17.

Theorem 18 (MV). For any $k \geq p-2$ and any $p > 3$,

$$F_{\{L_p,[k]\}}(x) =$$
$$= \sum_{i=0}^{p-3} i!x^i + x^{p-2} R_k(x) R_{k-1}(x) \sum_{a \in Q^{p-3}} N(a) \prod_{j=1}^{p-3} R_{k-j-a_j}(x).$$

As a generalization of Theorem 1 we get the following result.

Theorem 19 (MV). (i) For any m, $1 \leq m \leq k-2$,

$$F_{\{L_4,[k,m]\}}(x) = 1 + \frac{1}{4t^2} + \frac{U_{k-2}(t)U_{m-1}(t)}{tU_k(t)U_m(t)} + \frac{U_{k-m-2}(t)}{2tU_k(t)U_m(t)} \left(\frac{U_{k-m-2}(t)}{U_{k-m-1}(t)} + \frac{U_{k-m-3}(t)}{U_{k-m-2}(t)} \right), \qquad t = \frac{1}{2\sqrt{x}}.$$

(ii)

$$F_{\{L_4,[k,k-1]\}}(x) = F_{\{L_4,[k]\}}(x).$$

(6) 321-restricted permutations

The case of 321-restricted permutations is studied much less than the previous one. We start from the following generalization of Chow and West Theorem (i).

Theorem 20 ([MV3, Th. 1.2]). For $k \geq 2$ and any m, $1 \leq m \leq k-1$,

$$F_{\{321,[k,m]\}}(x) = R_k(x).$$

Therefore, the Wilf class of $\{132, [k,m]\}$ contains the pair $\{321, [k,m]\}$ as well. Once again, we have only an analytical proof of this result. In view of Theorems 1 and 2, it is a challenge to find a bijective proof.

A striking analog of Theorem 7 is given by the following result.

Theorem 21 ([Kr, Th. 10]). For $k \geq 3$ and any r, $1 \leq r \leq k$,

$$G^r_{321;[k,1]}(x) = \frac{U^{r-1}_{k-1}(t)}{(2t)^{r-1}U^{r+1}_k(t)}, \qquad t = \frac{1}{2\sqrt{x}}.$$

(7) other directions.

1. In [M1, Restricted 132-permutations and generalized patterns] Mansour considered the case of permutations avoiding (or containing once) 1-3-2 and avoiding (or containing once) arbitrary generalized pattern on k letters. For example,

Theorem(see [M1, Th. 2.3])

Let $\tau \in \{1\text{-}2, 2\text{-}1, 12, 21$. For any $k \geq 3$,

$$F_{\tau-3-\cdots k}(x) = R_k(x) = \frac{2tU_{k-1}(t)}{U_k(t)}, \quad t = \frac{1}{2\sqrt{x}}.$$

Another example, we say τ is a generalized wedge pattern if can be represented as $\tau = (\phi^1, \psi^1, \ldots, \phi^r, \psi^r)$ such that τ as classical is wedge pattern and each ϕ^j, ψ^j is either $a-(a+1)-\cdots-b$ or $a(a+1)-(a+2)-\cdots-b$.

Theorem(see [M1, Th 2.17])

For any generalized wedge pattern β on k letters,

$$F_\beta(x) = R_k(x).$$

2. For example, in [M2, Continued fractions and generalized patterns] Mansour considered the case of permutations avoiding 1-3-2 and containing a prescribed number of occurrences the pattern $12\text{-}3\text{-}\cdots\text{-}k$, or the pattern $21\text{-}3\text{-}\cdots\text{-}k$. For example,

Theorem(see [M2, Th.2.8])

Let $1 \leq r \leq k-2$ and $k \geq 3$. Then the generating function for the number of permutations containing $12\text{-}3\text{-}\cdots\text{-}k$ exactly r times in $S_n(1\text{-}3\text{-}2)$ is given by

$$\frac{x^{(r-1)/2}U_{k-2}^{r-1}\left(\frac{1}{2\sqrt{x}}\right)}{(1-x)U_k^{r+1}\left(\frac{1}{2\sqrt{x}}\right)}.$$

Theorem(see [M2, Th. 2.16])

Let $1 \leq r \leq k-2$ and $k \geq 3$. Then the generating function for the number of permutations containing $21\text{-}3\text{-}\cdots\text{-}k$ exactly r times in $S_n(1\text{-}3\text{-}2)$ is given by

$$\frac{x^{(r-1)/2}U_{k-2}^{r-1}\left(\frac{1}{2\sqrt{x}}\right)}{U_k^{r+1}\left(\frac{1}{2\sqrt{x}}\right)}.$$

What else, !!!!!!!!

New directions!!!!!!!!!

Other directions!!!!!!!!!

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