Discrete Morse Theory and applications

Combinatorics seminar

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Outline

- 1 Introduction
- 2 Preliminaries
- 3 Discrete morse theory
- 4 Generalized shellings and discrete morse functions
- 5 References

1 Introduction

- Discrete version of morse theory [Forman]
- Shellings [Björner, Wachs]
- Discrete morse theory and shellings [Chari]

1 Introduction

- Principle idea: construct a more efficient complex, while retaining topological properties
- For a given generalized shelling of a regular cell comples, there is a canonical discrete morse function

2 Preliminaries

- Σ finite regular cell complex
- $P(\Sigma)$ face poset wrt containment

  $\leq$ order, $\prec$ cover
- $\sigma \in \Sigma$, $\partial\sigma$ boundary subcomplex, $\overline{\sigma} = \sigma \cup \partial\sigma$

  $\sigma \cong \mathbb{B}^{\dim\sigma}$, $\partial\sigma \cong \mathbb{S}^{\dim\sigma-1}$, $\dim \Sigma = \max \dim_{\sigma \in \Sigma} \sigma$
- $\mathfrak{M}(\Sigma)$ maximal cells of Σ
- Σ pure if $\dim \sigma = d \quad \forall\ \sigma \in \mathfrak{M}(\Sigma)$

2 Preliminaries

- Shelling – *simplicial complex* version:

  Δ pure d-dimensional simplicial complex

  $F_1, ..., F_n$, ordering of $\mathfrak{M}(\Delta)$, shelling of Δ if $\forall i>1$:

    $F_i \cap (\cup_{j<i} F_j)$ nonempty union of (d-1)-faces of F_j

2 Preliminaries

- *Nonpure* shelling [Björner, Wachs]:

  $\sigma_1, ..., \sigma_m$, ordering of $\mathfrak{M}(\Sigma)$, shelling if

  $\dim \Sigma = 0$ or

  (i) $\exists$ ordering of $\mathfrak{M}(\partial\sigma_1)$ which is shelling

  (ii) $\partial\sigma_j \cap (\cup_{1\leq k \leq j-1} \partial\sigma_k)$ is pure and $(\dim \sigma_j\text{-}1)$-dimensional, for $2\leq j \leq m$

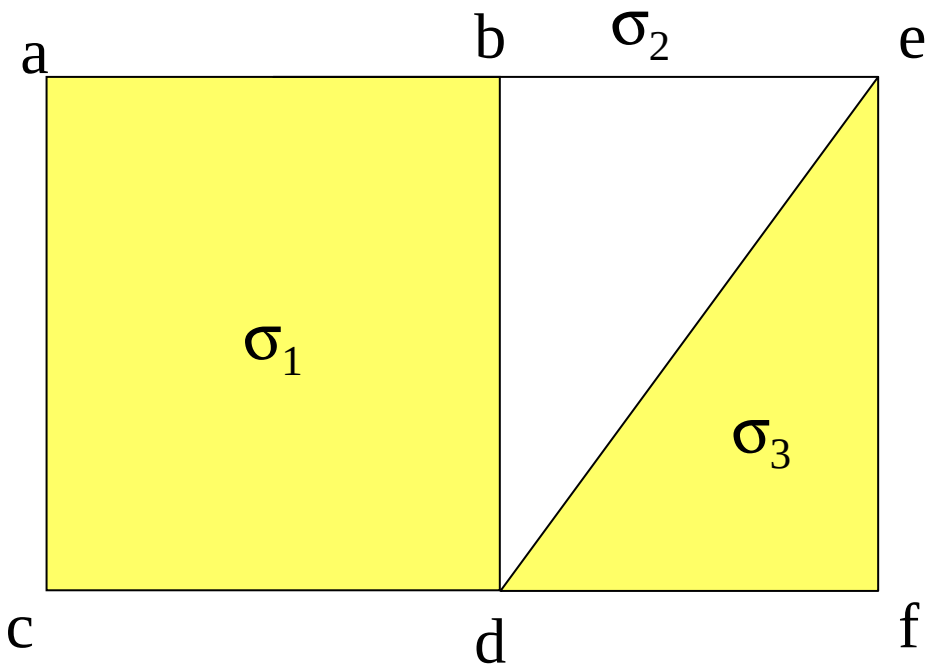
  (iii) $\exists$ ordering of $\mathfrak{M}(\partial\sigma_j)$ which is shelling where $\mathfrak{M}(\partial\sigma_j \cap (\cup_{1\leq k \leq j-1} \partial\sigma_k))$ appears first, for $2\leq j \leq m$

2 Preliminaries

- Theorem 2.1 [Björner, Wachs]

  Σ shellable $\Rightarrow \Sigma \simeq \bigvee \mathbb{S}^{d}$

2 Preliminaries



2 Preliminaries

- *Generalized* shelling [Chari]:

  $\sigma_1, ..., \sigma_m$, ordering of cells of Σ, generalized shelling if

  $\dim \Sigma = 0$ or

  (i) $\exists$ ordering of $\mathfrak{M}(\partial\sigma_1)$ which is shelling

  (ii) $\partial\sigma_j \cap (\cup_{1\le k \le j-1} \partial\sigma_k)$ is pure and $(\dim \sigma_j\text{-}1)$-dimensional, for $2\le j \le m$

  (iii) $\exists$ ordering of $\mathfrak{M}(\partial\sigma_j)$ which is shelling where $\mathfrak{M}(\partial\sigma_j \cap (\cup_{1\le k \le j-1} \partial\sigma_k))$ appears first, for $2\le j \le m$

  (iv) $\mathfrak{M}(\Sigma) \subseteq \{\sigma_1, ..., \sigma_m\}$

  (v) $\sigma_i \in \partial\sigma_j \Rightarrow i < j$

2 Preliminaries

(iv) $\mathfrak{M}(\Sigma) = \{\sigma_1, ..., \sigma_m\} \Rightarrow$

generalized shelling =

nonpure shelling

2 Preliminaries

- Proposition 2.1

  Σ simplicial complex, $F_1, ..., F_m$ ordered subset of faces:

  $F_1, ..., F_m$ generalized shelling

  $\Leftrightarrow$

  $\exists\ G_1, ..., G_m, G_i \subseteq F_i, \{ [G_i, F_i]\}_{1 \le i \le m}$ partitions Σ (S-partition) and $\cup_{1 \le i \le k}\ [G_i, F_i]$ simplicial complex for k= 1, .. , m

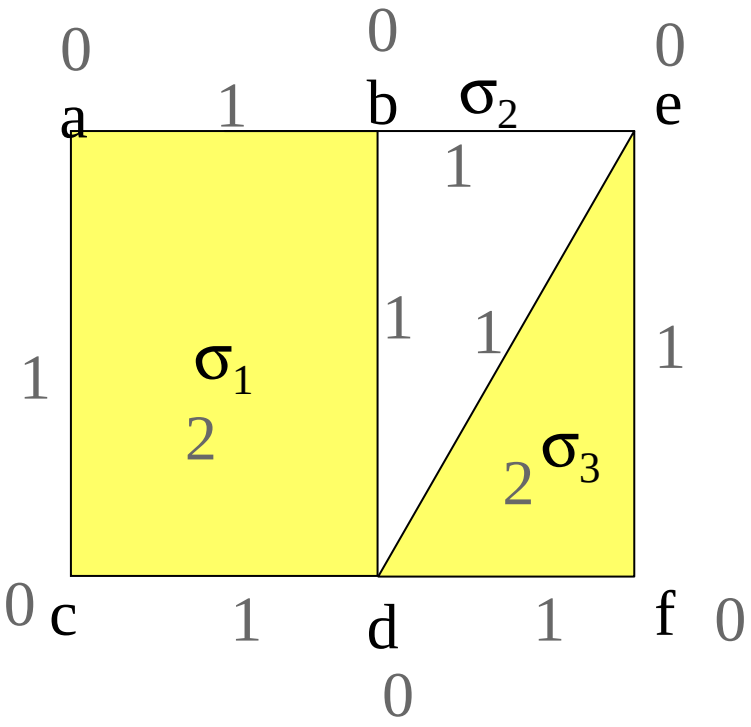
3 Discrete morse theory

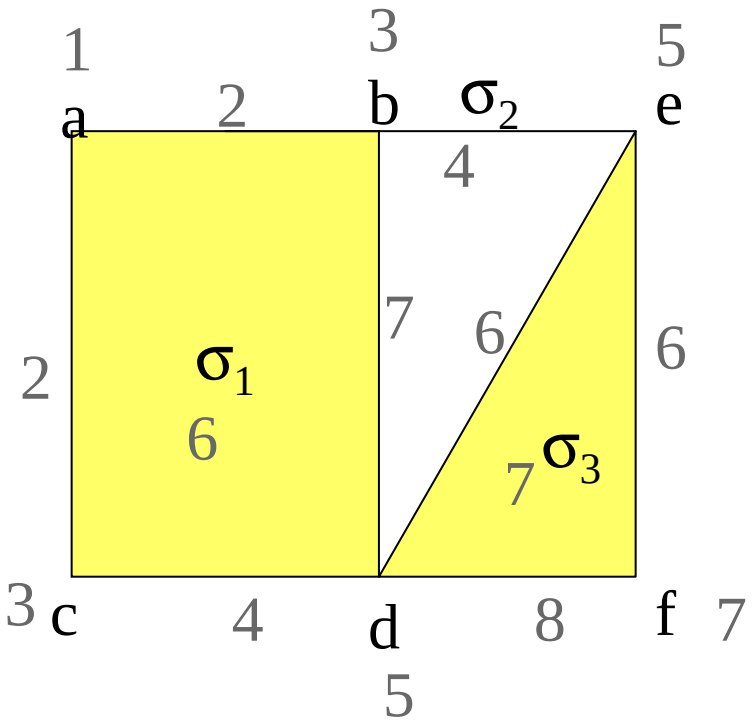
- f discrete morse function if
  $f : \Sigma \to \mathbb{R}$ and
  $\nu_B(\sigma) \leq 1$ and $\nu_C(\sigma) \leq 1 \quad \forall\ \sigma \in \Sigma$
  where $\nu_B(\sigma) = |\{\tau \in B(\sigma);\ f(\tau) \geq f(\sigma)\}|$
  $\nu_C(\sigma) = |\{\tau \in C(\sigma);\ f(\tau) \leq f(\sigma)\}|$

  ie almost increasing wrt dimension

  non interesting example: f = dim

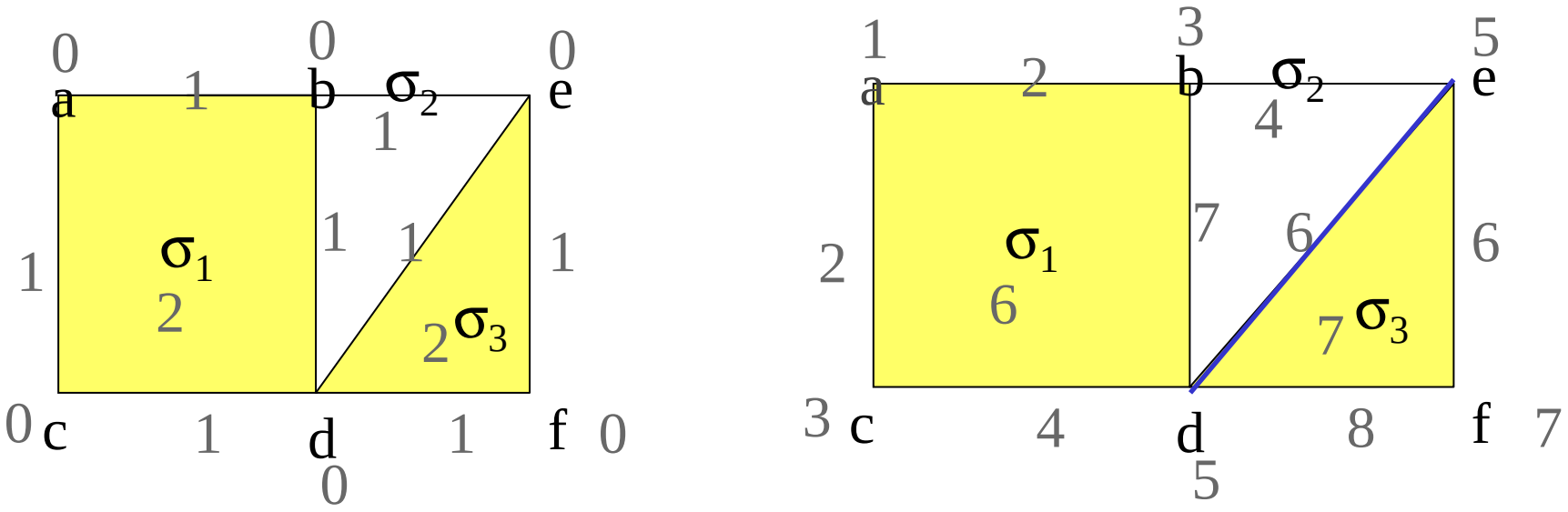
3 Discrete morse theory





3 Discrete morse theory

- σ *critical* cell if $\nu_B(\sigma) = \nu_C(\sigma) = 0$
- $C(f) = \{\sigma \in \Sigma; \nu_B(\sigma) = \nu_C(\sigma) = 0\}$



An efficient morse function has few critical cells

3 Discrete morse theory

- Theorem 3.1 [Forman]

  (i) $\Sigma \simeq C(f)$

  (ii) $\beta_j \leq m_j$

  where $m_j = |\{\sigma \in C(f); \dim \sigma = j\}|$

3 Discrete morse theory

- Lemma 3.2

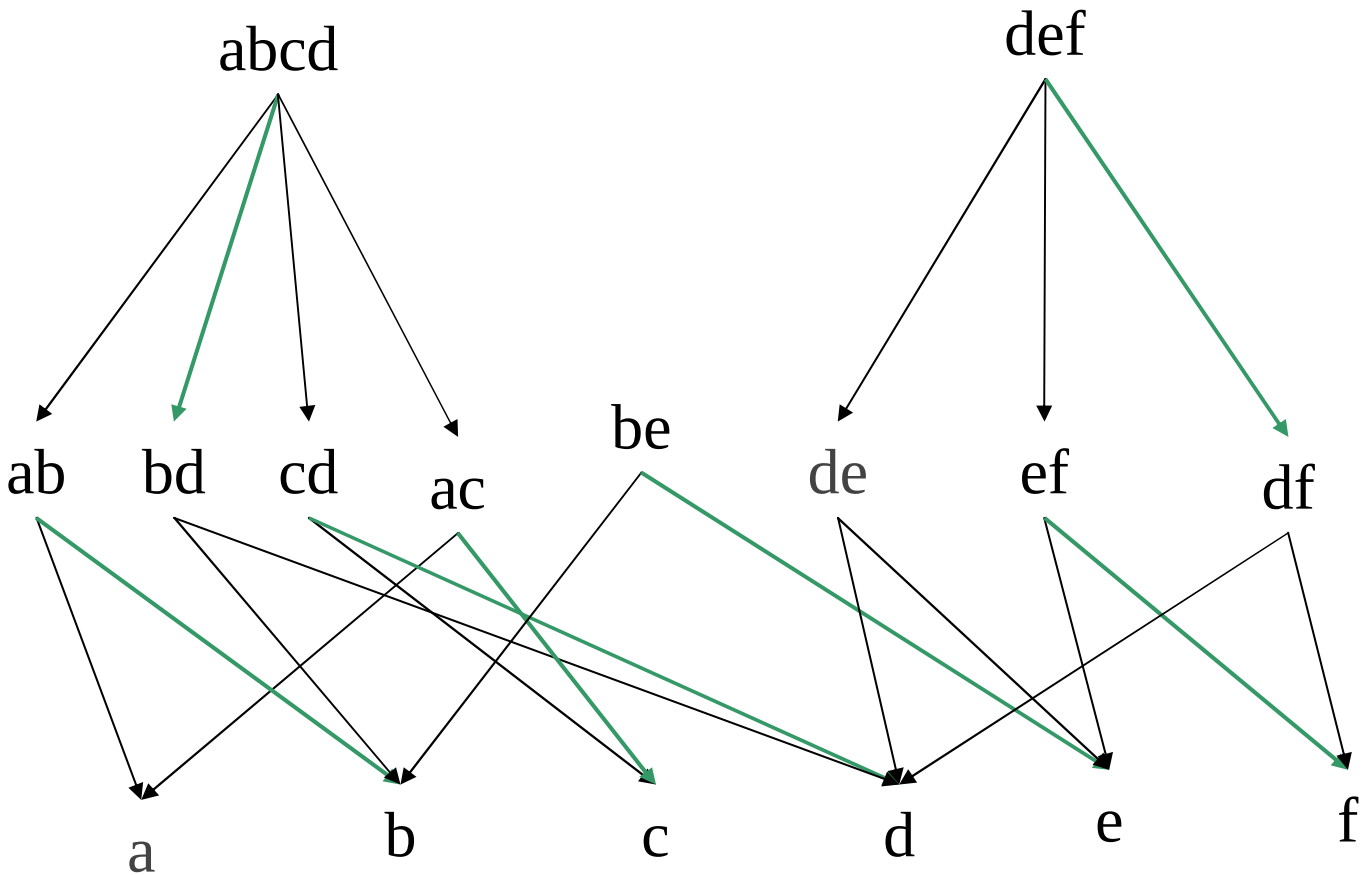
  $\nu_B(\sigma)\ \nu_C(\sigma) = 0$

3 Discrete morse theory

- G (Σ) hasse diagram of Σ

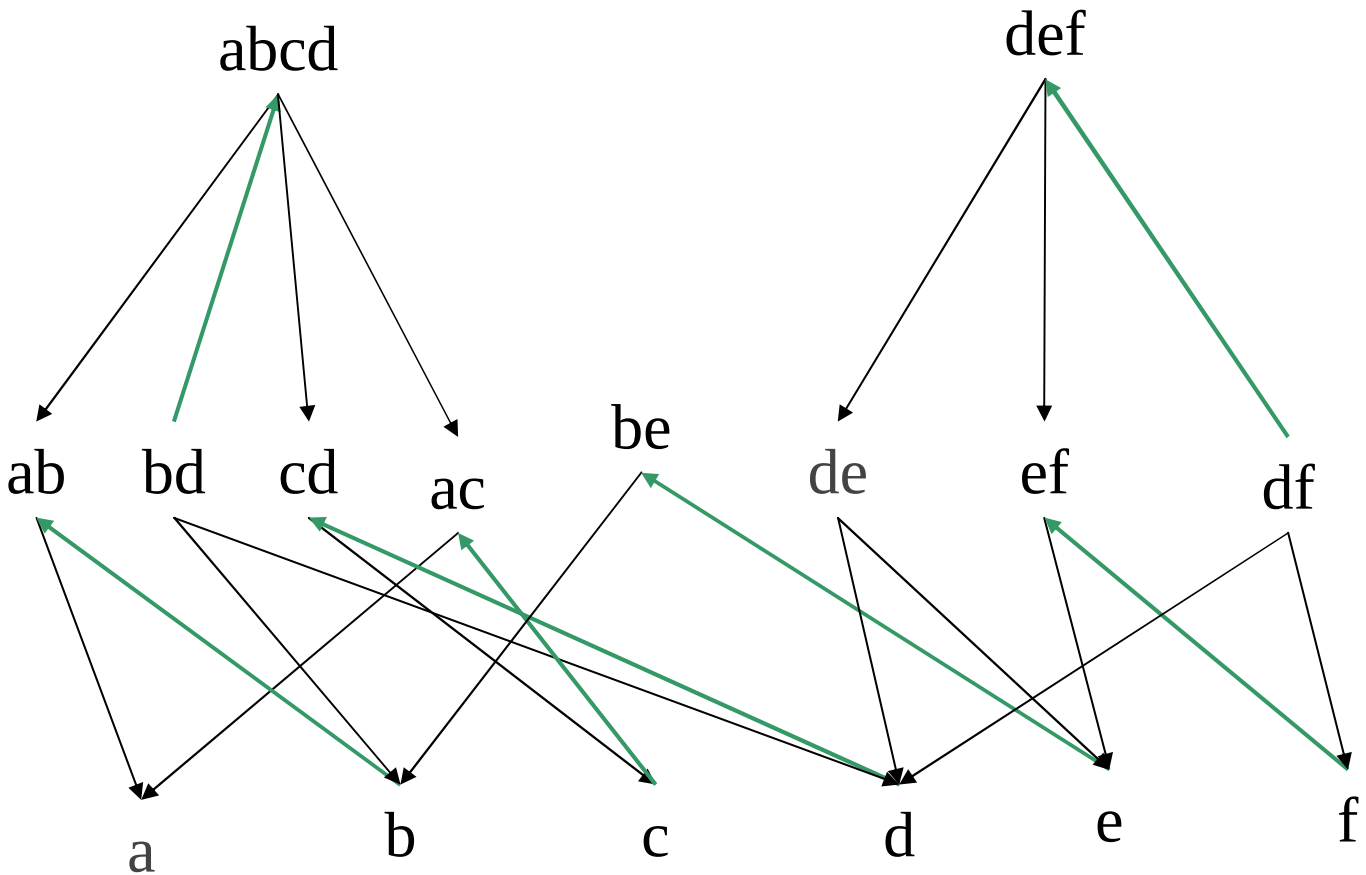
  directed acyclic
- ∃ M(f) avoids exactly C(f), ie cover relations where f in noincreasing wrt dimension

3 Discrete morse theory



$G_M(\Sigma) = G(\Sigma)$ reversing the edges in M

3 Discrete morse theory



$G_M(\Sigma) = G(\Sigma)$ reversing the edges in M

3 Discrete morse theory

- Proposition 3.3

  $C \subseteq \Sigma$

  $C = C(f)\ \exists\ f \Leftrightarrow\ \exists\ M$ on $G\ (\Sigma) : G_M\ (\Sigma)$ acyclic

  C avoids exactly M

4 Generalized shellings and morse functions

- d-pseudomanifold

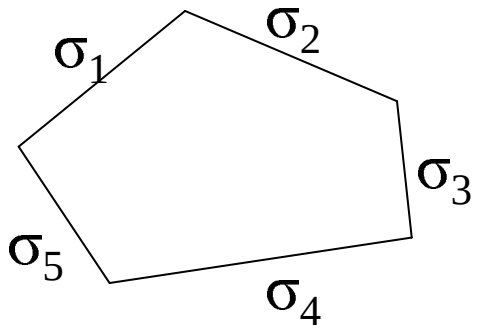
  pure d-dimensional regular cell complex

  (i) every (d-1)-cell is contained in at most two d-cells

  (ii) for any d-cells β, τ $\exists$ sequence of d-cells $\beta=\sigma_1, ..., \sigma_m=\tau$: σ_i, σ_{i+1} share (d-1)-cell

4 Generalized shellings and morse functions

- σ_j bounded if $\partial\sigma_j \cap (\cup_{1\leq k \leq j} \partial\sigma_k) = \partial\sigma_j$



σ_5 bounded

4 Generalized shellings and morse functions

- Proposition 4.1

  Σ d-pseudomanifold, $\sigma_1, \ldots, \sigma_m$ shelling, $v \in \overline{\sigma}_1$

  $\Rightarrow \Sigma$ admits morse function f:

  (i) Σ d-sphere $\Rightarrow C(f) = \{ v, \sigma_m \}$

  Σ d-ball $\Rightarrow C(f) = \{ v \}$

  (ii) restricted to $\cup_{1 \leq k \leq j} \overline{\sigma}_k$ $C(f) = \{ v \}$, $j<m$

4 Generalized shellings and morse functions

- Theorem 4.2

  $\sigma_1, ..., \sigma_m$ generalized shelling, $v \in \overline{\sigma}_1$
  $\Rightarrow \exists$ f : v critical

  σ critical $\Leftrightarrow$ σ bounded

4 Generalized shellings and morse functions

- Corollary 4.3

  Σ d-dimensional

  m_j = | j-dimensional bounded cells in generalized shelling |

  $\Rightarrow$

  (i) $\Sigma \simeq$ cell complex with m_0+1 points and
  m_j j-dimensional cells for j=1, .., d

  (ii) the bounded cells appear in non-increasing order of dimension $\Rightarrow \Sigma \simeq \vee_{0\leq j\leq d} \mathbb{S}^{m_j}$

References

- Chari (2000) *On discrete morse functions and combinatorial decompositions* Discrete Mathematics 217 101-113
- Forman (2002) *A user's guide to discrete morse theory* Séminaire Lotharingien 48
- Forman (accepted) *How many equilibria are there? An introduction to morse theory*