Cyclic pattern avoidance

Antoine Vella

Cyclic modifications of pattern occurrence

"Wrap around" in the:

- - position

  $\underline{7}\ 1\ \underline{6}\ 5\ \underline{2}\ 4 \mid \underline{3}\ \underline{9}\ 8$

  - values

  $6\ \underline{4}\ \underline{1}\ 9\ \underline{8}\ 2\ \underline{7}\ 5\ \underline{3}$

  "hybrid" — cyclic arrangements

- both

  $\underline{8}\ 2\ \underline{7}\ 5\ \underline{3} \mid 6\ \underline{4}\ \underline{1}\ 9$

  "purely cyclic" — orbits

- neither

  $8\ \underline{3}\ \underline{9}\ 4\ \underline{7}\ 5\ \underline{6}\ 1\ \underline{2}$

  "classical" — permutations

All examples are occurrences of 25431

Formal definition

Ordered set: $(X, \mathcal{T}_X)$, with $\mathcal{T}_X \subseteq X^k$, $k \in \mathbb{N}$

D, E, F, G ordered sets, $f : D \to E$, $g : F \to G$ arbitrary functions

$$\begin{array}{ccc} D & \xrightarrow{f} & E \\ \alpha \downarrow & & \uparrow \beta \\ F & \xrightarrow{g} & G \end{array}$$

$$f \sim g \iff \left\{ \begin{array}{c} \exists \text{ order-isomorphisms } \alpha, \beta \\ \alpha : D \longleftrightarrow F,\ \beta : g(F) \longleftrightarrow f(D) \\ \text{s.t. } \ f = \alpha \circ g \circ \beta \end{array} \right.$$

Subsets of ordered sets inherit the orders.

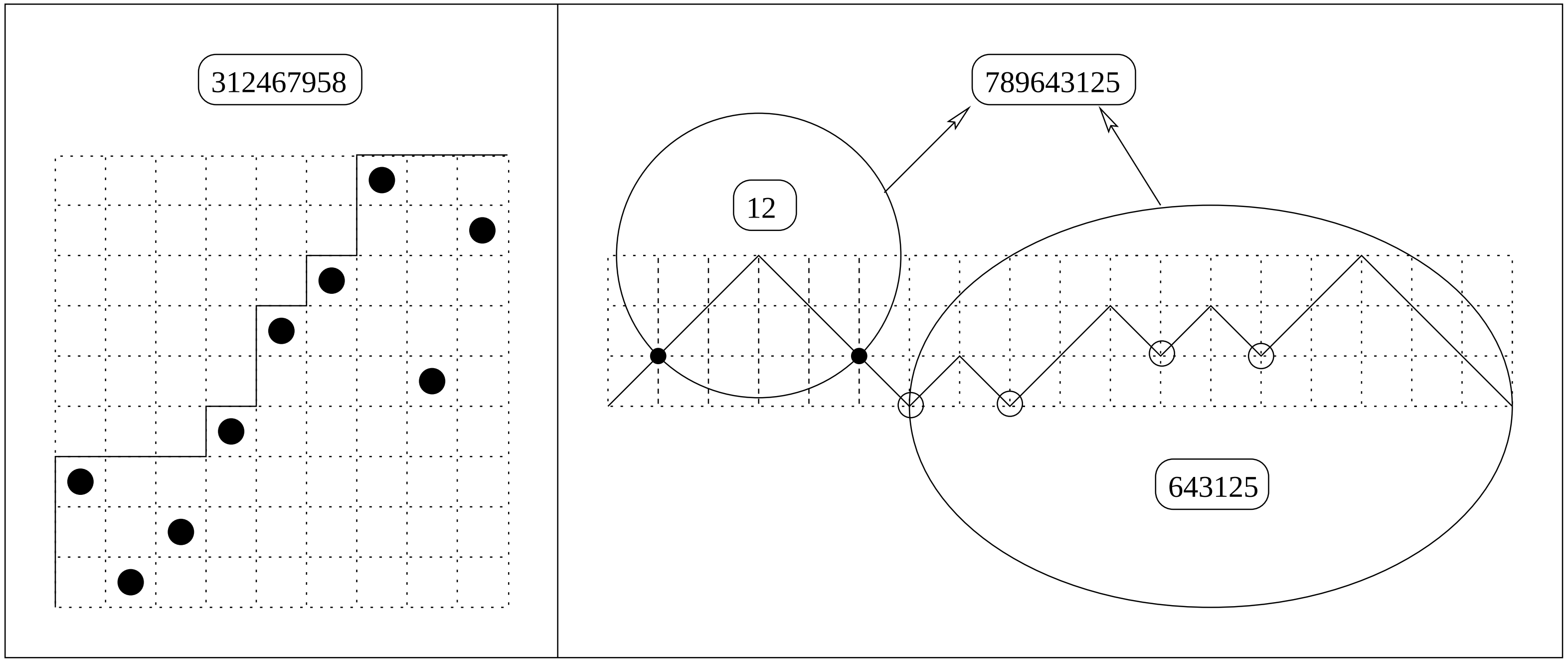
An *occurrence* of h in f is a subset $S \subseteq D$ s.t. $f|_S \sim h$.

Overview

$$
\begin{array}{llll}
\text{classical (simultaneous)} & 321, 2143 & 132, 3241 & \\
 & \uparrow & \uparrow & \\
\text{hybrid} & (4321) & (1324) & (1243) \\
 & \wr & & \\
\text{purely cyclic} & ((4321)) & ((1243))^* &
\end{array}
$$

* trivial

The bijections of Deutsch and Krattenthaler I



Sink-or-float bijection

First-return bijection

The bijections of Deutsch and Krattenthaler II

- First-return bijection:
  - panoramic Dyck paths $\longleftrightarrow$ 132-avoiding permutations
  - valleys $\longleftrightarrow$ descents
  - **THM** non-decreasing Dyck paths $\longleftrightarrow$ $\{132, 3241\}$-avoiding

- Sink-or-float bijection:
  - escalating Dyck paths $\longleftrightarrow$ 321-avoiding permutations
  - **THM** simple Dyck paths $\longleftrightarrow$ $\{321, 2143\}$-avoiding

Non-decreasing Dyck paths

Theorem: Under the first-return bijection of panoramic Dyck paths to 132-avoiding permutations, non-decreasing Dyck paths correspond bijectively to those permutations which also avoid 3241.

Proposition: The subsets of cardinality $2k$ of $[n+k-1]$ are in one-to-one correspondence with non-decreasing Dyck paths of length n with precisely k valleys.

Corollary: The number of non-decreasing Dyck paths of length n with k valleys ($\{132, 3241\}$-avoiding permutations with k descents) is

$$\binom{n+k-1}{2k}.$$

Corollary: Non-decreasing Dyck paths ($\{132, 3241\}$-avoiding permutations) are enumerated by the Fibonacci numbers of odd index.

Statistics on 321-avoiding permutations I

Via the sink-or-float bijection:

- peaks $\longleftrightarrow$ sufficiencies (left-to-right maxima)
  - hilltops $\longleftrightarrow$ fixed points
  - mountain tops $\longleftrightarrow$ excedances

Narayana Distributions: sufficiencies, left-to-right maxima, excedances.

$$\begin{aligned}\mathsf{FIX}\,|\,\mathsf{EXC}_{\mathcal{Z}}(x,y,z) &= \mathsf{HILL}\,|\,\mathsf{MNT}(x,y,z)\\ &= \frac{2L-K-\sqrt{K^2-4x}}{2(L^2-LK+x)}\end{aligned}$$

where $L = 1 + x - xy,\ K = 1 + x - xz.$

Statistics on 321-avoiding permutations II

Via the sink-or-float bijection:

- long horizontal edges $\longleftrightarrow$ descents
- long vertical edges $\longleftrightarrow$ dips

$$
\begin{aligned}
\mathsf{DES}\,|\,\mathsf{IDES}_{\mathcal{Z}}(x,y,z) &= \mathsf{LHOR}\,|\,\mathsf{LVER}(x,y,z) \\
&= \frac{Q-\sqrt{Q^2-4P}}{2P}
\end{aligned}
$$

where
$$
\begin{aligned}
P &= x(1-x+xy)(1-x+xz) \\
Q &= 1-x^2(y-1)(z-1).
\end{aligned}
$$

Statistics on 321-avoiding permutations III

Extracting the coefficients:

Proposition: The number of 321-avoiding permutations of length n with precisely p descents and q dips is given by:

$$\sum_{i\geq 0} c_i \sum_{\substack{2s+a_1=n\\ p_1+p_2=p\\ q_1+q_2=q}} (-1)^{n+p+q+i}\binom{i}{p_1}\binom{i}{q_1}\binom{s}{p_2}\binom{s}{q_2}\binom{2i+s}{s}\binom{2i-p_1-q_1}{a_1-p_1-q_1-i}.$$

This is actually computationally useful: e.g. of the

1583850964596120042686772779038896

321-avoiding permutations of length 60, there are

2539791795216418415246700

which have precisely 19 descents and 5 dips.

The descent statistic on 321-avoiding permutations

Specializing $z = 1$ and extracting coefficients:

Proposition: The number of 321-avoiding permutations of length n with precisely m descents is given by:

$$\sum_i (-1)^{n+i+m} \frac{1}{i+1} \binom{2i}{n} \binom{n}{i} \binom{n-i}{m}$$

$$= \sum_i (-1)^{n+i+m} c_i \binom{i}{n-i} \binom{n-i}{m}.$$

A refinement of the Catalan numbers

n	m 0	1	2	3	4	5	6	7	Row sum
2	1	1							2
3	1	4							5
4	1	11	2						14
5	1	26	15						42
6	1	57	69	5					429
7	1	120	252	56					1430
8	1	247	804	364	14				4862
9	1	502	2349	1800	210				16796
10	1	1013	6455	7515	1770	42			58786
11	1	2036	16962	27940	11055	792			208012
12	1	4083	43086	95458	57035	8217	132		742900
13	1	8178	106587	305812	257257	62062	3003		2674440
14	1	16369	258153	931385	1049685	381381	37037	429	9694845

- $\mathrm{Des}(2,n) = 2^{n-3}\ n(n-5) + \binom{n+1}{2}$
- $\mathrm{Des}(3,n) = \frac{1}{3}2^{n-6}\ n(n-1)(n^2-13n+46) - \binom{n+1}{3}$
- $\mathrm{Des}(4,n) = \frac{1}{9}2^{n-10}\ n(n-1)(n-2)(n^3-24n^2+203n-612) + \binom{n+1}{4}$.

Simple Dyck Paths

Definition: An escalating Dyck path is *simple* if it has at most one long vertical edge or at most one long horizontal edge.

Theorem: Under the sink-or-float bijection of escalating Dyck paths to 321-avoiding permutations, simple Dyck paths correspond bijectively to those permutations which also avoid 2143.

Corollary: The $\{321, 2143\}$-avoiding permutations are precisely the Grassmannian permutations and their inverses. The number of such permutations of length n is $2^{n+1} - \binom{n+1}{3} - 2n - 1$.

Cyclic arrangements: the pattern (1243) I

The wraparound map: $\mathcal{P}([n]) \to \mathcal{S}_{n+1}$

$$
\begin{array}{ccccccccc}
 & \mathbf{1} & \mathbf{2} & 3 & 4 & 5 & \mathbf{6} & 7 & 8 \\
\hline
 & & & 5 & 4 & 3 & & 2 & 1 \\
 & & & \longleftarrow & \longleftarrow & \longleftarrow & & \longleftarrow & \longleftarrow \\
6 & T & T & B & B & B & T & B & B \\
 & \longrightarrow & \longrightarrow & & & & \longrightarrow & & \\
 & 7 & 8 & & & & 9 & &
\end{array}
$$

$\{1, 2, 6\} \subseteq [1, 8] \mapsto$ 678543921

Cyclic arrangements: the pattern (1243) II

Definition: A *bisecting* subset of $[n]$ is a set of the form $[k]$, with $k < n$.

Theorem: The wraparound map takes the set of non-bisecting subsets of $[n]$ bijectively onto the set of (1243)-avoiding cyclic arrangements of $[n+1]$.

Corollary: The number of (1243)-avoiding cyclic arrangements of $[n]$ is $2^{n-1} - n + 1$. They are in a one-to-one correspondence with the Grassmannian permutations of $[n-1]$.

Cyclic arrangements: the patterns (1324) and (4321)

Proposition: A permutation σ avoids 132 and 3241 if and only if $\widehat{\sigma}$ avoids (1324)

Corollary: The (1324)-avoiding cyclic arrangements of $[n]$ are enumerated by the Fibonacci numbers of odd index. They are in one-to-one correspondence with non-decreasing Dyck paths of length $n-1$.

Proposition: A permutation σ avoids 123 and 3412 if and only if $\widehat{\sigma}$ avoids (1234)

Corollary: The number of (1234)-avoiding cyclic arrangements of $[n]$ is

$$2^n - \binom{n}{3} - 2n + 1.$$

They are in one-to-one correspondence with the simple Dyck paths of length $n-1$.

Cyclic operators

$\mathcal{F}(r,s)$ denotes the set of functions from $[r]$ to $\mathbb{Z}_s$.

Proposition: A non-trivial operator $\langle a,b\rangle \in \mathcal{G}_n$ has at least one fixed point if and only if $\gcd(a,n) = \gcd(b,n)$. In this case, let $k = \gcd(a,n)$ and $p = n/k$; the fixed points of $\langle a,b\rangle$ are in one-to-one correspondence with the set $\mathcal{S}_k \times \mathcal{F}(k,p)$. Moreover, there exist $\mu, \nu \in \mathbb{Z}_p^*$ such that $a = \mu k, b = \nu k$ and the fixed points of $\langle a,b\rangle$ are precisely the fixed points of $< k, \lambda k >$, where $\lambda = \lceil \mu^{-1}\nu \rfloor_p$.

Corollary: The total number of orbits of length n is given by

$$\frac{1}{n^2}\sum_{kp=n} \phi(p)^2 \; k! \; p^k$$

where ϕ is the Euler totient function.

Affine permutations

For $k|n$, let $\mathcal{A}_k^{(n)}$ denote the set of solutions of

$$\sigma(x \oplus k) = \sigma(x) \oplus \lambda k$$

for some λ.

Proposition: For positive integers k, p, consider the group $\mathcal{H}_{k,p}$ with ground set $S_k \times \mathcal{F}(k,p) \times \mathbb{Z}_p^*$ and group operation, denoted by $\odot$, given by

$$(\alpha, c, \lambda) \odot (\beta, d, \mu) = (\alpha \circ \beta, c + \lambda^{-1}(\sigma \circ d), \lambda\mu).$$

If $n = kp$, $\mathcal{A}_k^{(n)}$ is a subgroup of S_n isomorphic to $\mathcal{H}_{k,p}$.

Orbits: The pattern ((4321)) I

The following are equivalent:

- σ is ((4321))-avoiding;
- σ is (4321)-avoiding;
- σ' is $\{321, 2143\}$-avoiding;
- σ' is the image under the sink-or-float bijection of a simple Dyck path;
- σ' is Grassmannian or monodipic.

Orbits: The pattern ((4321)) II

Lemma: Suppose $k|n$ and the operator $\langle k, \lambda k\rangle \in \mathcal{G}_n$ has a ((4321))-avoiding fixed point corresponding to a horizontally simple Dyck path. Then $\lambda \in \{1, 2\}$.

Proposition: If $n = kp$ and p is odd, the number of Grassmannian permutations (horizontally simple Dyck paths) which correspond to ((4321))-avoiding fixed points of $\langle k, 2k\rangle$ is 2^{k-1}. If σ is such a permutation, then $\bar{\sigma}$ is itself Grassmannian.

Orbits: The pattern ((4321)) III

Lemma: If $n = kp$, $\lambda \in \mathbb{Z}_p^*$ and the operator $\langle k, \lambda k \rangle$ has at least one ((4321))-avoiding fixed point, then precisely one of the following holds:

I $\lambda = 1$ and all the ((4321))-avoiding fixed points of $\langle k, \lambda k \rangle$ correspond to the identity permutation;

II $\lambda = 2$ and all the ((4321))-avoiding fixed points of $\langle k, \lambda k \rangle$ correspond to Grassmannian, but not monodipic, permutations (horizontally simple, but not vertically simple, Dyck paths);

III $\lambda = \lceil 2^{-1} \rfloor_p$ and all the ((4321))-avoiding fixed points of $\langle k, \lambda k \rangle$ correspond to monodipic, but not Grassmannian, permutations (vertically simple, but not horizontally simple, Dyck paths);

IV $p = 3$ and $\lambda = 2 = \lceil 2^{-1} \rfloor_p$.

Orbits: The pattern ((4321)) IV

Theorem: The number of $((4321))$-avoiding orbits of $[n]$ is given by:

$$\frac{1}{n}\left(2^n - \binom{n}{3} - n - \underline{\widehat{\frac{2n}{3}}} + \sum_{\substack{2 \nmid p | n \\ 1 < p \leq n}} \phi(p) 2^{\frac{n}{p}}\right)$$

where ϕ is the Euler totient function and $\underline{\widehat{x}}$ denotes x if x is an integer, and 0 otherwise.